# ON ESTIMATION OF REGULARITY FOR GAUSSIAN PROCESSES

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Abstract. We consider a real Gaussian process $X$ with unknown smoothness $r_0$ where $r_0$ is a nonnegative integer and the mean-square derivative $X^{(r_0)}$ is supposed to be locally stationary of index $\beta_0$. From $n+1$ equidistant observations, we propose and study an estimator of $(r_0, \beta_0)$ based on results for quadratic variations of the underlying process. Various numerical studies of these estimators derive their properties for finite sample size and different types of processes, and are also completed by two examples of application to real data.

## 1. Introduction

For a real stationary and non differentiable Gaussian process with covariance $\mathbb{K}(s,t) = \mathbb{K}(|t-s|, 0)$ such that $\mathbb{K}(t,0) = \mathbb{K}(0,0) - A|t|^{2\beta_0} + o(|t|^{2\beta_0})$ as $|t| \to 0$, the parameter $\beta_0$, $0 < \beta_0 < 1$, is closely related to the fractal dimension of the sample paths. This relationship is developed in particular in the works of Adler (1981) and Taylor and Taylor (1991) and it gave rise to an important literature around the estimation of $\beta_0$. We refer in particular to Constantine and Hall (1994), Istas and Lang (1997) and Kent and Wood (1997) for estimators based on quadratic variations and their extensions. Still in this stationary Gaussian framework, Chan et al. (1995) introduce a periodogram-type estimator whereas Feuerverger et al. (1994) use the number of level crossings.

In this paper, our aim consists in the estimation of the couple $(r_0, \beta_0)$ when the Gaussian process is supposed to be $r_0$-times differentiable with a locally stationary $r_0$-th derivative of regularity $\beta_0$. Thus, we generalise the previous works in two ways:

(1) the process has an unknown regularity $r_0$ to be estimated as well as the smoothness $\beta_0$,
(2) the process is not supposed to be stationary not even with stationary increments.

Our methodology is based both on the estimator of $r_0$, say $\widehat{r_0}$, proposed by Blanke and Vial (2011) and on Kent and Wood (1997) for the estimation of $\beta_0$. we give a new result concerning the mean square error of $\widehat{r_0}$ as well as almost sure rates of convergence of the global regularity index $\widehat{H} = 2(\widehat{r_0} + \widehat{\beta_0})$. As an application, we provide plugged-estimators, based on lagrange piecewise interpolation, which are optimal in rate, for the relative problems of integration and approximation of a discretised sample path.

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## 2. Methodology

2.1. **Framework.** Let $X = \{X(t),\, t \in [0,T]\}$ be a real Gaussian process observed at the $(n+1)$ equally spaced times $i\delta_n$, $i = 0, \ldots, n$ for some positive sequence $\delta_n$ such that $\delta_n \to 0$ and $n\delta_n \to T > 0$. This process is supposed to satisfy the following assumptions.

**Assumption 2.1** (A2.1)**.** *$X$ has $r_0$, $r_0 \in \mathbb{N}_0$, continuous derivatives in quadratic mean with $X^{(r_0)}$ supposed to be locally stationary:*

$$\lim_{h\to 0} \sup_{s,t\in[0,T],|s-t|\le h, s\neq t} \left| \frac{\mathbb{E}\big(X^{(r_0)}(s) - X^{(r_0)}(t)\big)^2}{|s-t|^{2\beta_0}} - d_0(t) \right| = 0 \tag{2.1}$$

*where $\beta_0 \in ]0,1[$ and $d_0$ is a positive continuous function on $[0,T]$.*

Here, local stationarity makes reference to Berman (1974)'s meaning. Assumption A2.1 implies in particular that the covariance function $\mathbb{K}(s,t) = \mathrm{Cov}\,(X(s), X(t))$ is also continuously differentiable with $K^{(r,r)}(s,t) = \mathrm{Cov}\,(X^{(r)}(s), X^{(r)}(t))$, for $r = 0, \ldots, r_0$. Also, the mean of the process $\mu(t) := \mathbb{E}X(t)$ is a $r_0$-times continuously differentiable function with $\mathbb{E}X^{(r)}(t) = \mu^{(r)}(t)$, $r = 0, \ldots, r_0$.

In this paper, we propose and study an estimator of the Hölder regularity index $H = 2(r_0 + \beta_0)$, say $\widehat{H}$. To this aim, we need the following additional condition:

**Assumption 2.2** (A2.2($p$))**.** *For either $p = 1$ or $p = 2$, $\mathbb{K}^{(r_0+p,r_0+p)}(s,t)$ exists on $[0,T]^2 \backslash \{s = t\}$ and satisfies for some $D_p > 0$:*

$$\left|\mathbb{K}^{(r_0+p,r_0+p)}(s,t)\right| \le D_p\, |s-t|^{-(2p-2\beta_0)}\,.$$

*Moreover, we suppose that the mean $\mu(\cdot)$ admits a continuous derivative of order $(r_0+1)$ on $[0,T]$.*

2.2. **Specific examples.** A first important example of process meeting our conditions is the $r_0$-fold integrated fractional Brownian motion. For $r_0 = 0$, $X = W_{\beta_0}$ is the fractional Brownian motion (i.e. with covariance $\mathbb{K}(s,t) = \frac{1}{2}(s^{2\beta_0} + t^{2\beta_0} - |s-t|^{2\beta_0})$). For $r_0 \ge 1$, we have

$$X(t) = \int_0^t \int_0^{u_{r_0}} \int_0^{u_{r_0-1}} \cdots \int_0^{u_2} W_{\beta_0}(u_1)\,\mathrm{d}u_1\,\mathrm{d}u_2 \cdots \mathrm{d}u_{r_0}, \quad r_0 \ge 1$$

One gets $d_0(t) \equiv 1$, and for $\beta_0 \neq \frac{1}{2}$, $D_1 = \beta_0\,|2\beta_0 - 1|$ while $D_2 = \beta_0\,|2\beta_0 - 1| \ldots |2\beta_0 - 3|$. For special case of the Wiener process ($\beta_0 = \frac{1}{2}$), we have $\mathbb{K}^{(r_0+1,r_0+1)}(s,t) \equiv 0$ on $[0,1]^2 \backslash \{s = t\}$.

Another examples of processes may be found among those satisfying the conditions of Sacks and Ylvisaker of order $r_0$ ($r_0 \ge 0$). For $\Omega_+ = \{(s,t) \in ]0,1[^2 \colon s > t\}$ and $\Omega_- = \{(s,t) \in ]0,1[^2 \colon s < t\}$:

(a) $\mathbb{K} \in C^{r_0,r_0}([0,1]^2)$, on $\Omega_+ \cup \Omega_-$ $L = \mathbb{K}^{(r_0,r_0)}$ has partial derivatives up to order $p$, with either $p = 1$ or $p = 2$, continuous and continuously extendible to $\overline{\Omega_+}$ and $\overline{\Omega_-}$ by continuity.

(b) $L_-^{(1,0)}(s,s) - L_+^{(1,0)}(s,s) = \alpha > 0$, $s \in [0,1]$ where $L_j^{(1,0)}$ is the extension of $L^{(1,0)}$ over $[0,1]^2$, continuous on $\overline{\Omega_j}$ and on $[0,1]^2 \backslash \overline{\Omega_j}$ with $j \in \{-,+\}$.

In this case, $\beta_0 = 1/2$ with $d_0(t) \equiv \alpha$ and these processes satisfy A 2.2(p) for $p = 1, 2$.

Another class of processes are stationary ones with spectral density $\varphi$ such that: $\varphi(\lambda) = c_0(1 + c_1\lambda^2)^{-\gamma}$ where $\gamma > \frac{1}{2}$, $c_0 > 0$, $c_1 > 0$. Then for $\gamma - \frac{1}{2} \notin \mathbb{N}$, one gets $r_0 = \left[\gamma - \frac{1}{2}\right]$ and $\beta_0 = \gamma - \frac{1}{2} - r_0$, and A 2.2(p), $p = 1, 2$ is satisfied at least for $\gamma \in \mathbb{N}$. A typical process is the Ornstein-Uhlenbeck one, with $\varphi(\lambda) = \frac{1}{2\pi}(\theta^2 + \lambda^2)^{-1}$ and $\mathbb{K}(s,t) = (2\theta)^{-1}\exp(-\theta\,|s-t|)$, so $\gamma = 1$, $r_0 = 0$, $\beta_0 = \frac{1}{2}$ and $d_0(t) \equiv 1$. Following Lasinger (1993), one may construct stationary processes with $r_0 \geq 1$ by $r_0$-fold repeated integration.

Since all previous processes are stationary or with stationary increments, the function $d_0$ is reduced to a constant. Of course, cases with non constant $d_0(\cdot)$ may be also obtained. In particular, we have processes with a smooth enough trend belong to our general class :

**Lemma 2.1.** *Let $Y$ be a zero mean process with given regularity $(r_0, \beta_0)$ and asymptotic function $d_0(t) \equiv C_{r_0,\beta_0}$. For a positive function $a \in C^{r_0+p}([0,1])$ and $m \in C^{r_0+p}([0,1])$ $(p = 1, 2)$, if $X(t) = a(t)Y(t) + m(t)$, then $X$ has regularity $(r_0, \beta_0)$ with asymptotical function $D_{r_0,\beta_0}(t) = a^2(t)C_{r_0,\beta_0}$ and $X$ satisfies A2.2(p).*

2.3. **Estimation of $\beta_0$.** Kent and Wood (1997) and Constantine and Hall (1994) use "dilation" quadratic variations to estimate the fractal index for stationary Gaussian process, not differentiable in quadratic mean. Following this methodology, we define a global estimator of $H = 2(r_0 + \beta_0)$ which is based on dilated quadratic variations and a consistent preliminary estimator of $r_0$.

First for integers $r$, $\overline{r}$ such that $\overline{r} \geq r \geq 1$ and $u \in \mathbb{N}^*$, we consider the $u$-dilated quadratic variations of $X$ of order $r$:

$$\Delta_{r,k}^{(u)} X = \sum_{i=0}^{\overline{r}} a_{i,r} X((iu+k)\delta_n), \quad k = 0, \ldots, n - u\overline{r} \tag{2.2}$$

where order $r$ of the sequence $(a_{i,r})$ means:

$$\sum_{i=0}^{\overline{r}} a_{i,r} = 0, \;\; \sum_{i=1}^{\overline{r}} a_{i,r} i^r \neq 0, \quad \text{and if } r \geq 2, \;\; \sum_{i=1}^{\overline{r}} a_{i,r} i^p = 0 \text{ for } p = 1, \ldots, r-1.$$

Also, for $r = r_0, r_0 + 1$, we assume that:

$$\sum_{i=0}^{\overline{r}} \sum_{j=0}^{\overline{r}} a_{i,r} a_{j,r} \,|i - j|^H \neq 0.$$

Note that, in the case where $u = 1$ and $\delta_n = n^{-1}$, $\Delta_r^{(1)}$ represents the classical finite difference of order $r$. A typical example of sequence $(a_{j,r})$ is given by $a_{j,r} = \binom{r}{j}(-1)^{r-j}$, with $\overline{r} = r$, for which one obtains, in particular, $\sum_{j=0}^{r} a_{j,r} j^r = r!$. Another examples are furnished by Daubechies wavelets $D(2p)$ of order $r = p$ with vanishing first moments of order 0 to $p - 1$, and such that $\overline{r} + 1 = 2p$ (see Daubechies, 1992).

Asymptotic properties of $\Delta_{r,k}^{(u)} X$ are important in the comprehension of our considered estimators. From now on, we set: $\overline{\big(\Delta_r^{(u)} X\big)^2} = \dfrac{\sum_{k=0}^{n_r} \big(\Delta_{r,k}^{(u)} X\big)^2}{n_r+1}$ with $n_r := n - u\overline{r}$.

**Proposition 2.1.** *Under Assumptions A2.1, one obtains:*

*(i) for $r = r_0 + 1, r_0 + 2$:*

$$\delta_n^{-H}\, \mathbb{E}\Big( \overline{\big(\Delta_r^{(u)} X\big)^2} \Big) \xrightarrow[n\to\infty]{} u^H \ell(r, r_0, \beta_0)$$

*where* $\ell(r, 0, \beta_0) = -\dfrac{1}{2} \displaystyle\sum_{i,j=0}^{\overline{r}} a_{i,r} a_{j,r} |i-j|^H \frac{1}{T} \int_0^T d_0(t)\, \mathrm{d}t$ *while if* $r_0 \geq 1$,

$$\ell(r, r_0, \beta_0) = \frac{(-1)^{r_0+1} T^{-1} \int_0^T d_0(t)\, \mathrm{d}t}{2(2\beta_0 + 2r_0) \cdots (2\beta_0 + 1)} \sum_{i,j=0}^{\overline{r}} a_{i,r} a_{j,r} |i-j|^H . \tag{2.3}$$

*(ii) for $r_0 \geq 1$ and $r = 1, \dots, r_0$:*

$$\delta_n^{-2r}\, \mathbb{E}\Big( \overline{\big(\Delta_r^{(u)} X\big)^2} \Big) \xrightarrow[n\to\infty]{} u^{2r} \ell(r) \tag{2.4}$$

*with*

$$\ell(r) = \Big( \frac{\sum_{i=0}^{\overline{r}} i^r a_{i,r}}{r!} \Big)^2 \frac{1}{T} \int_0^T \mathbb{E}\big(X^{(r)}(t)\big)^2\, \mathrm{d}t.$$

Proposition 2.1 implies that a good choice of $r$ ($r = r_0+1$, $r_0+2$) could provide an estimate of $H$, at least with an adequate combination of $u$-dilated quadratic variations of $X$. To this end, we propose a two steps procedure:

- We consider the consistent estimate of $r_0$, say $\widehat{r_0}$ derived in Blanke and Vial (2011) and based on $\Delta_{r,k}^{(1)} X$ :

$$\widehat{r_0} = \min\Big\{ r \in \{2, \dots, m_n\} : n^{2r-2} \overline{\big(\Delta_r^{(1)} X\big)^2} \geq b_n \Big\} - 2. \tag{2.5}$$

  If the above set is empty, we fix $\widehat{r_0} = l_0$ for an arbitrary value $l_0 \notin \mathbb{N}_0$. Here, $m_n \to \infty$ but if an upper bound $B$ is known for $r_0$, one has to choose $m_n = B+2$. The threshold $b_n$ is a positive sequence chosen such that : $n^{-2(1-\beta_0)} b_n \to 0$ and $n^{2\beta_0} b_n \to \infty$ for all $\beta_0 \in ]0, 1[$. For example, omnibus choices are given by $b_n = (\ln n)^\alpha$, $\alpha \in \mathbb{R}$.
- Next, we derive two family of estimators for $H$, namely $\widehat{H}_n^{(p)}$, with either $p = 1$ or $p = 2$:

$$\widehat{H}_n^{(p)} := \widehat{H}_n^{(p)}(u, v) = \frac{\ln\Big( \overline{\big(\Delta_{\widehat{r_0}+p}^{(u)} X\big)^2} \Big) - \ln\Big( \overline{\big(\Delta_{\widehat{r_0}+p}^{(v)} X\big)^2} \Big)}{\ln(u/v)}$$

  where $u, v$ ($u < v$) are given integers, chosen by the statistician. Then, two estimators $\widehat{\beta_0}^{(p)}$ of $\beta_0$ can be directly derived:

$$\widehat{\beta_0}^{(p)} = (\widehat{H}_n^{(p)} - 2\widehat{r_0})/2, \quad p = 1, 2.$$

**Remark 2.1.** *For $r_0 = \widehat{r_0} = 0$, $u = 1$ and $v = 2$, $\widehat{\beta_0}^{(p)}$ corresponds to the ordinary least squares estimators defined in Constantine and Hall (1994) (with the choice $p = 1$) and Kent and Wood (1997) ($p = 2$). Note that new estimators may be derived with other choices of $(u, v)$ such as $(u, v) = (1, 4)$ which seems to perform well, see Section 3.2.*

In Blanke and Vial (2011), an exponential bound is obtained for $\mathbb{P}(\widehat{r_0} \neq r_0)$ implying that, almost surely for $n$ large enough, $\widehat{r_0}$ is equal to $r_0$. Here, we complete this result with a bound for the mean square error of $\widehat{r_0}$.

**Theorem 2.1.** *Under assumption A2.1 and A2.2(p), $p = 1$ or $p = 2$, we have*

$$\mathbb{E}(\widehat{r_0} - r_0)^2 = \mathcal{O}\Big(m_n^3 \exp\big(-\varphi_n(p)\big)\Big)$$

*where $\varphi_n(p)$ is defined by*

$$\varphi_n(p) = C_1(r_0) \times \begin{cases} n\mathbb{1}_{]0,\frac{1}{2}[}(\beta_0) + n(\ln n)^{-1}, \mathbb{1}_{\{\frac{1}{2}\}}(\beta_0) + n^{2-2\beta_0}\mathbb{1}_{]\frac{1}{2},1[}(\beta_0) \text{ if } p = 1 \\ n \text{ if } p = 2 \end{cases}$$

*for some positive constant $C_1(r_0)$.*

Remark that one may choose $m_n$ tending to infinity as slowly as wanted. Indeed, the unique restriction is that $r_0$ belongs to the grid $\{1, \dots, m_n\}$ for large enough $n$. From a practical point of view, one may choose a preliminary fixed bound $B$, and, in the case where the estimator return the non-integer value $l_0$, replace $B$ by $B'$ greater than $B$.

The bias of $\widehat{H}_n^{(p)}$ will be controlled by a second-order condition of local stationarity, more specifically we have to strengthen the relation (2.1) in:

$$\lim_{h\to 0} \sup_{s,t\in[0,T],|s-t|\le h, s\neq t} \left| |s-t|^{-\beta_1} \Big( \frac{\mathbb{E}\big(X^{(r_0)}(s) - X^{(r_0)}(t)\big)^2}{|s-t|^{2\beta_0}} - d_0(t) \Big) - d_1(t) \right| = 0 \tag{2.6}$$

for a positive $\beta_1$ and continuous function $d_1$.

**Theorem 2.2.** *If relation (2.6) and assumption A2.2(p) are satisfied, we obtain*

$$\limsup_{n\to\infty} V_n^{(p)} \left| \widehat{H}_n^{(p)} - H \right| \le C_2(p) \quad a.s.$$

*where $C_2(p)$ is some positive constant and*

$$V_n^{(1)} = \min\Big(n^{\beta_1}, \sqrt{\frac{n}{\ln n}}\mathbb{1}_{]0,\frac{3}{4}[}(\beta_0) + \frac{\sqrt{n}}{\ln n}\mathbb{1}_{\{\frac{3}{4}\}}(\beta_0) + \frac{n^{2(1-\beta_0)}}{\ln n}\mathbb{1}_{]\frac{3}{4},1[}(\beta_0)\Big),$$

$$V_n^{(2)} = \min\Big(n^{\beta_1}, \sqrt{\frac{n}{\ln n}}\Big).$$

**Remark 2.2.** *Of course, since* $n\delta_n \xrightarrow[n\to\infty]{} T > 0$*, the previous result can be expressed in terms of* $\delta_n$ *with:*

$$V_n^{(1)} = \min\Big(\delta_n^{-\beta_1}, \frac{1}{\sqrt{\delta_n \ln(\delta_n^{-1})}}\mathbb{1}_{]0,\frac{3}{4}[}(\beta_0) + \frac{1}{\sqrt{\delta_n}\ln(\delta_n^{-1})}\mathbb{1}_{\{\frac{3}{4}\}}(\beta_0) + \frac{\delta_n^{-2(1-\beta_0)}}{\ln(\delta_n^{-1})}\mathbb{1}_{]\frac{3}{4},1[}(\beta_0)\Big),$$

$$V_n^{(2)} = \min\Big(\delta_n^{-\beta_1}, \frac{1}{\sqrt{\delta_n \ln(\delta_n^{-1})}}\Big).$$

## 3. Applications and numerical results

All the numerical results are obtained on simulation of trajectories using two different methods : for stationary processes or with stationary increments we use the procedure described in Wood and Chan (1994) and for CARMA (continuous ARMA) processes, we use Tsai and Chan (2000). Each of them consists in $n$ equally spaced observation points on $[0, 1]$ and 1000 simulated sample paths. All computations have been performed with the R software (R Core Team, 2012).

3.1. **Simulation study : estimation of** $r_0$**.** This section is dedicated to the numerical properties of two estimators of $r_0$. We consider the estimator introduced by Blanke and Vial (2011), defined in (2.5). Another estimator was introduced in Blanke and Vial (2008) and is based on Lagrange interpolator polynomials, says $\tilde{r}_n$. More precisely, for $\delta_n = n^{-1}$ et $T = 1$, it is defined by

$$\tilde{r}_n = \min\Big\{r \in \{1, \dots, m_n\} : \frac{1}{r\widetilde{n}_r}\sum_{k=0}^{r\widetilde{n}_r-1}\Big(X\Big(\frac{2k+1}{n}\Big) - \widetilde{X}_r\Big(\frac{2k+1}{n}\Big)\Big)^2 \geq n^{-2r}b_n\Big\} - 1$$

where $\widetilde{n}_r = \left[\frac{n}{2r}\right]$ and $\widetilde{X}_r(s)$ is defined for all $s \in [0, 1]$ and each $r \in \{1, \dots, m_n\}$ in the following way : there exist $k = 0, \dots, \widetilde{n}_r - 1$ such that $s \in \mathcal{I}_k := [\frac{2kr}{n}, \frac{2(k+1)r}{n}]$ and the interpolates of $X(s)$ is defined by

$$\widetilde{X}_r(s) = \sum_{i=0}^{r} L_{i,k,r}(s)X\Big(\frac{2(kr+i)}{n}\Big), \text{ with } L_{i,k,r}(s) = \prod_{\substack{j=0\\ j\neq i}}^{r}\frac{(s - \frac{2(kr+j)}{n})}{(\frac{2(kr+i)}{n} - \frac{2(kr+j)}{n})}.$$

Both estimators use a critical value depending on $n$, here we choose $b_n = 1/\ln n$, due to convergence properties. Table 3.1 illustrates the strong convergence of both estimators and shows that this convergence is valid even for small number of observation points $n$, up to 10 for the estimator $\widehat{r_0}$. We may noticed that, for misspecified estimations our estimators overestimate the number of derivatives. Remark also that, for identical sample paths, $\widehat{r_0}$ seems to be more robust than $\tilde{r}_n$. This behaviour was expected as the latter uses only half of the observations for the detection of the jump in the quadratic mean. In these first results, processes have fractal index $\beta_0$ equals to 1/2, but alternative choices of $\beta_0$ are of interest, so we consider the fractional Brownian motion (in short fBm) and the integrated fractional Brownian motion (in short ifBm), with respectively $\widehat{r_0} = 0$ and $\widehat{r_0} = 1$ and $\beta_0 \in ]0, 1[$. Table 3.2 shows that $\widehat{r_0}$ succeeds in estimating the regularity $r_0 = 0$ even if the process is near regularity $r_0 = 1$, of course the number of observations must be sufficient and even more

TABLE 3.1. Value of the empirical probability that $\widehat{r_0}$ or $\tilde{r}_n$ equals $r_0$ or $r_0+1$ with $n = 10$ or 25.

| | Wiener process, $r_0 = 0$ | | CARMA(2,1), $r_0 = 1$ | | CARMA(3,1), $r_0 = 2$ | |
|---|---|---|---|---|---|---|
| | Number of equally spaced observations $n$ | | | | | |
| event | 10 | 25 | 10 | 25 | 10 | 25 |
| $\tilde{r}_n = r_0$ | 0.995 | 1.000 | 0.913 | 1.000 | 0.585 | 0.999 |
| $\tilde{r}_n = r_0 + 1$ | 0.005 | 0.000 | 0.087 | 0.000 | 0.415 | 0.001 |
| $\widehat{r_0} = r_0$ | 1.000 | 1.000 | 1.000 | 1.000 | 0.999 | 1.000 |
| $\widehat{r_0} = r_0 + 1$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.001 | 0.000 |

TABLE 3.2. Value of the empirical probability that $\widehat{r_0}$ or $\tilde{r}_n$ equals $r_0$ for a fractional Brownian motion or an integrated fractional Brownian motion of fractal index $2\beta_0$.

| | $\widehat{r_0} = r_0$ | | | | | $\tilde{r}_n = r_0$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | number of equally spaced observations $n$ | | | | | | | | | |
| | 50 | 100 | 500 | 1000 | 1200 | 50 | 100 | 500 | 1000 | 1200 |
| fBm $\beta_0$ | | | | | | | | | | |
| 0.90 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.655 | 0.970 | 1.000 | 1.000 | 1.000 |
| 0.95 | 0.969 | 0.999 | 1.000 | 1.000 | 1.000 | 0.002 | 0.002 | 0.004 | 0.134 | 0.331 |
| 0.97 | 0.242 | 0.521 | 1.000 | 1.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 0.98 | 0.019 | 0.015 | 0.0420 | 0.5258 | 0.759 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| ifBm $\beta_0$ | | | | | | | | | | |
| 0.02 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.90 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.000 | 0.000 | 0.645 | 0.999 | 1.000 |
| 0.95 | 0.305 | 0.888 | 1.000 | 1.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 0.97 | 0.000 | 0.000 | 0.292 | 0.993 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

important for increasing values of $r_0$ nd $\beta_0$. This latter result is clearly apparent when one compares the errors obtained for an ifBm with $\beta_0 = 0.95$ and a fBm with $\beta_0 = 0.95$. Finally, we can see once more that $\tilde{r}_n$ is less robust against increasing $\beta_0$ while it appears that, for $n = 2000$ and each simulated path, $\widehat{r_0}$ is able to distinguish processes with regularity $(0, 0.98)$ and $(1, 0.02)$, a quite indiscernible difference.

3.2. **Simulation study: estimation of $H$ and $\beta_0$.** This part is dedicated to the numerical properties of estimators $\widehat{H}_n^{(p)}$, for $p = 1$ or 2 using the values $u = 1$ and $v = 4$ (giving a smaller variance than $u = 1$ and $v = 2$). It ends with real data examples.

For the numerical part, we focus on the study of fBm, ifBm and CARMA(3,1). Table 3.3 illustrates the performance of our estimators when $\beta_0$, $r_0$ are increasing by computing the empirical mean-square error from our 1000 simulated sample paths and number of equally spaced observation $n = 1000$. It appears that, contrary to $\widehat{H}_n^{(2)}$, the estimator $\widehat{H}_n^{(1)}$ slightly deteriorates for values of $\beta_0$ greater than 0.8. This result is in agreement with the rate of convergence established in Theorem 2.2 which depends on $\beta_0$. The bias is negative and seems to be unsensitive to the value of $r_0$ but the mean-square error is slightly deteriorated from

TABLE 3.3. Values of mean square error and bias (between brackets) for estimators $\widehat{H}_n^{(p)}$, for $p = 1$ or 2 and $n = 1000$.

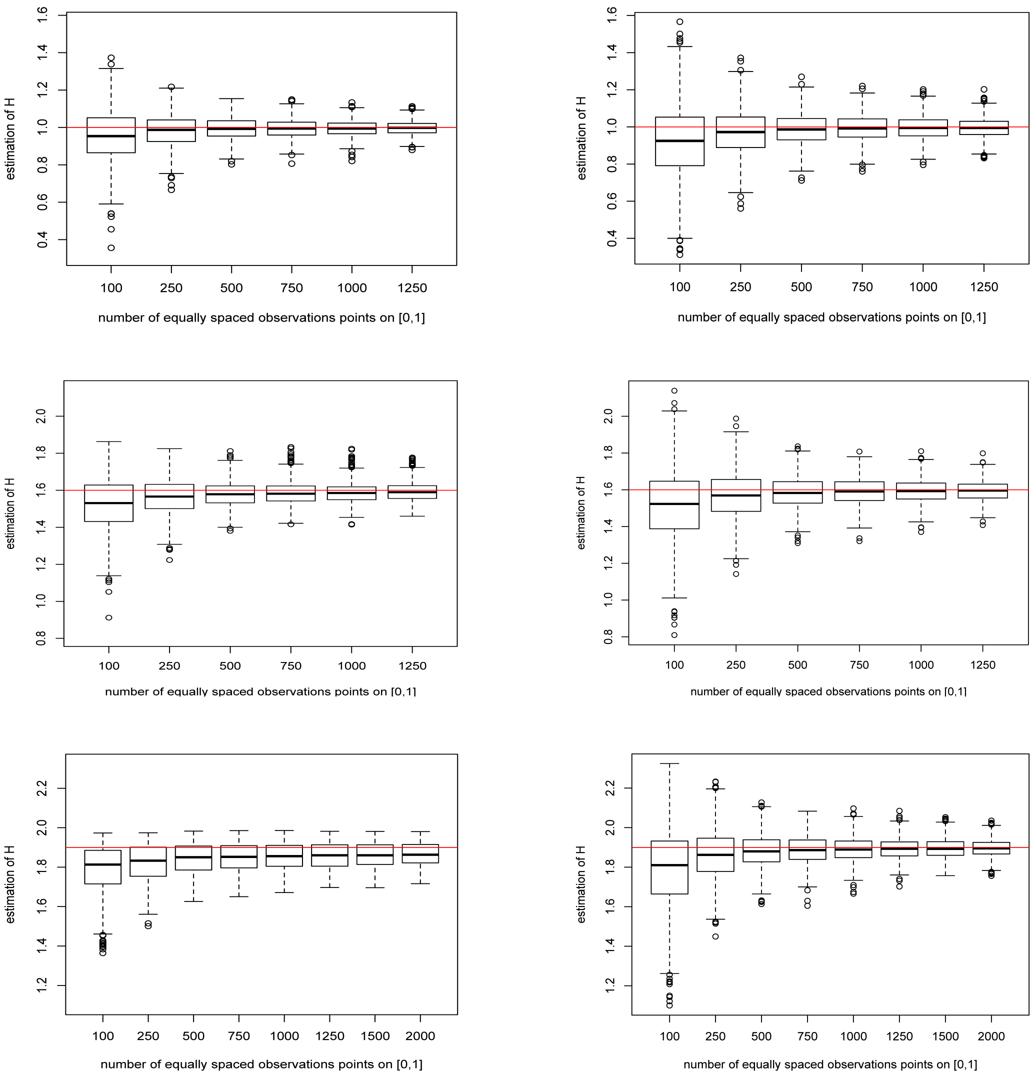
| | fBm $\beta_0$ 0.2 | 0.5 | 0.8 | 0.9 | 0.95 |
|---|---|---|---|---|---|
| $\widehat{H}_n^{(1)}$ | 0.0017 (−0.0020) | 0.0019 (−0.0045) | 0.0034 (−0.0120) | 0.0054 (−0.0295) | 0.0065 (−0.0448) |
| $\widehat{H}_n^{(2)}$ | 0.0030 (−0.0026) | 0.0039 (−0.0038) | 0.0040 (−0.0057) | 0.0039 (−0.0069) | 0.0039 (−0.0084) |
| | **ifBm $\beta_0$ 0.2** | **0.5** | **0.8** | **0.9** | **0.95** |
| $\widehat{H}_n^{(1)}$ | 0.0032 (−0.0072) | 0.0026 (−0.0047) | 0.0041 (−0.0150) | 0.0061 (−0.0342) | 0.0073 (−0.0488) |
| $\widehat{H}_n^{(2)}$ | 0.0055 (−0.0106) | 0.0051 (−0.0060) | 0.0046 (−0.0060) | 0.0044 (−0.0061) | 0.0043 (−0.0061) |

$r_0 = 0$ to $r_0 = 1$ in both cases. Finally, for $\beta_0 < 0.8$, $H_n^{(1)}$ seems preferable to $\widehat{H}_n^{(2)}$, possibly due to a lower variance of this estimator. Nevertheless, both estimators perform globally well on these numerical experiments.

Results of Theorem 2.2 are also illustrated in Table 3.4 where we have computed the regression of $\ln(\mathbb{E}|\widehat{H}_n^{(p)} - H|)$ on $\ln n$ for various values of $n$ and $\mathbb{E}|\widehat{H}_n^{(p)} - H|$ estimated from our 1000 simulated sample paths. As expected, the slope (corresponding to our arithmetical rate of decay) is constant and approximatively equal to 0.5 for $\widehat{H}_n^{(2)}$ while, for $\widehat{H}_n^{(1)}$, the decrease is apparent for high values of $\beta_0$. Finally, Figure 3.1 illustrates the behaviour of the estimators $\widehat{H}_n^{(p)}$ with $p = 1$ or 2, for different values of the regularity parameter $\beta_0$. As we can see, boxplots are similar for both estimators when $\beta_0 = 0.5$ or $\beta_0 = 0.8$ but with a quite larger dispersion for $\widehat{H}_n^{(2)}$. For $\beta_0 = 0.95$, $\widehat{H}_n^{(2)}$ clearly outperforms $\widehat{H}_n^{(1)}$ from $n = 500$ observations. Estimation appears more difficult for smaller values of $n$, but it is quite typical in our considered framework.

Next, Table 3.5 illustrates the impact of estimating $H$ when the order $r$ in quadratic variation is misspecified. Kent and Wood (1995), section 7, have already noticed that orders $r > 1$ are relevant for smooth processes. In fact estimating $H$ requires the knowledge of $r_0$ or an upper bound of it. On the other hand, working with a too high value of $r_0$ may induce artificial variability in estimation, so a precise estimation of $r_0$ is important. Here, our numerical results show that, if the order $r$ of quadratic variation used for estimating $H$ is less than $r_0 + 1$, then the quantity estimated is $2r$ and not $H$.

All previous examples are local stationary with a constant function $d_0$. Processes meeting our conditions but with no stationary increments may be constructed with Lemma 2.1. As an example, from $Y$ a standard Wiener process ($r_0 = 0$, $\beta_0 = 0.5$) or an integrated one ($r_0 = 1$, $\beta_0 = 0.5$), we simulate $X(t) = (t^{r_0+0.7} + 1)\,Y(t)$ having the regularity $(r_0, 0.5)$, Figure 3.2 illustrates a Wiener sample path and its transformation. Results are summarized in Table 3.6. Comparing with Table 3.3, it appears that the estimation is only slightly damaged for $r_0 = 1$ but of the same order when $r_0 = 0$. Non-stationary processes may also be obtained by adding some smooth trend. To this aim, we used same sample paths as in Table 3.3 with the additional trend $m(t) = (1 + t)^2$, see Figure 3.3. We may noticed in




$\widehat{H}_n^{(1)}$ $\widehat{H}_n^{(2)}$

FIGURE 3.1. Each boxplot correspond to 1000 estimation of $H$ by $\widehat{H}_n^{(1)}$ on the left and $\widehat{H}_n^{(2)}$ on the right of the graph. Each realization consists in $n$ equally spaced observations on $[0,1]$ of a fBm with $\beta_0 = 0.5$ (top), $\beta_0 = 0.8$ (middle), $\beta_0 = 0.95$ (bottom), where $n = 100, 250, 500, 750, 1000, 1250, 1500, 2000$.

Table 3.7 that we obtain exactly the same results for the estimator $\widehat{H}_n^{(2)}$ and that a slight damage is observed for $\widehat{H}_n^{(1)}$.

Let us turn to examples based on real data sets. We first focus on roller height data introduced by Laslett (1994), which consists in $n = 1150$ heights measure at 1 micron intervals along a drum of a roller. This example was already studied in Kent and Wood (1997), they noticed that local self similarity may hold at sufficiently fine scales, so the

TABLE 3.4. Rates of convergence illustrated by linear regression for n in $\{500, 750, 1000, 1250\}$.

| | $\widehat{H}_n^{(1)}$ | | $\widehat{H}_n^{(2)}$ | |
|---|---|---|---|---|
| | slope | $R^2$ | slope | $R^2$ |
| fBm $\beta_0 = 0.5$ | -0.488 | 0.998 | -0.489 | 0.995 |
| $\beta_0 = 0.6$ | -0.475 | 0.998 | -0.488 | 0.995 |
| $\beta_0 = 0.7$ | -0.426 | 0.994 | -0.489 | 0.997 |
| $\beta_0 = 0.8$ | -0.334 | 0.989 | -0.491 | 0.997 |
| $\beta_0 = 0.9$ | -0.225 | 0.990 | -0.495 | 0.997 |
| $\beta_0 = 0.95$ | -0.186 | 0.995 | -0.503 | 0.997 |
| ifBm $\beta_0 = 0.9$ | -0.302 | 0.987 | -0.561 | 0.999 |
| $\beta_0 = 0.95$ | -0.244 | 0.978 | -0.559 | 0.999 |

TABLE 3.5. Mean value and standard deviation (between brackets) of the estimator $\widehat{H}_n^{(p)}$ based on quadratic variation of order $r_0$ or $r_0 - 1$ instead of $\widehat{r_0} + 1$ or $\widehat{r_0} + 2$.

| | Order $r_0 - 1$ | | Order $r_0$ | |
|---|---|---|---|---|
| | number $n$ of equidistant observations | | | |
| | 100 | 500 | 100 | 500 |
| ifBm $\beta_0 = 0.2$ | | | 1.903 (0.064) | 1.961 (0.025) |
| $\beta_0 = 0.5$ | | | 1.955 (0.035) | 1.992 (0.006) |
| $\beta_0 = 0.8$ | | | 1.966 (0.024) | 1.994 (0.004) |
| CARMA(3,1) | 1.970 (0.0140) | 1.994 (0.003) | 3.919 (0.058) | 3.985 (0.0109) |

TABLE 3.6. Value of MSE and bias (between brackets) for non constant $d_0(\cdot)$.

| | Wiener | Integrated Wiener |
|---|---|---|
| $\widehat{H}_n^{(1)}$ | 0.0021 $(-0.0032)$ | 0.0032 (0.0061) |
| $\widehat{H}_n^{(2)}$ | 0.0043 $(-0.0042)$ | 0.0058 $(-0.0091)$ |

TABLE 3.7. Value of MSE and bias (between brackets) for estimators $\widehat{H}_n^{(p)}$, for $p = 1$ or 2 in presence of a smooth trend.

| | fBm | | ifBm | |
|---|---|---|---|---|
| | $\beta_0 = 0.5$ | $\beta_0 = 0.8$ | $\beta_0 = 0.5$ | $\beta_0 = 0.8$ |
| $\widehat{H}_n^{(1)}$ | 0.0022 (0.0151) | 0.0283 (0.1525) | 0.0027 (0.0083) | 0.0141 (0.0863) |
| $\widehat{H}_n^{(2)}$ | 0.0039 $(-0.0038)$ | 0.0040 $(-0.0057)$ | 0.0051 $(-0.0060)$ | 0.0046 $(-0.0060)$ |

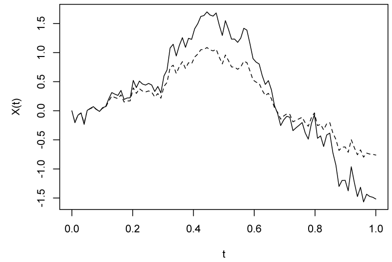



FIGURE 3.2. fBm with $\beta_0 = 0.5$ (solid) and its transformation (dashed) used in Table 3.6.

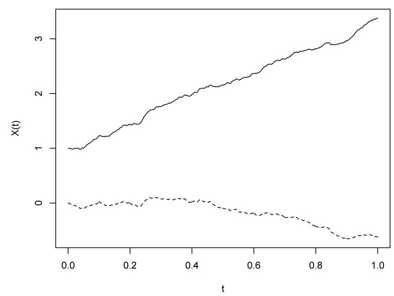



FIGURE 3.3. Sample path of a fBm with $\beta_0 = 0.8$ (dashed line) and the same with a trend $m(t) = (1+t)^2$ (solid line).

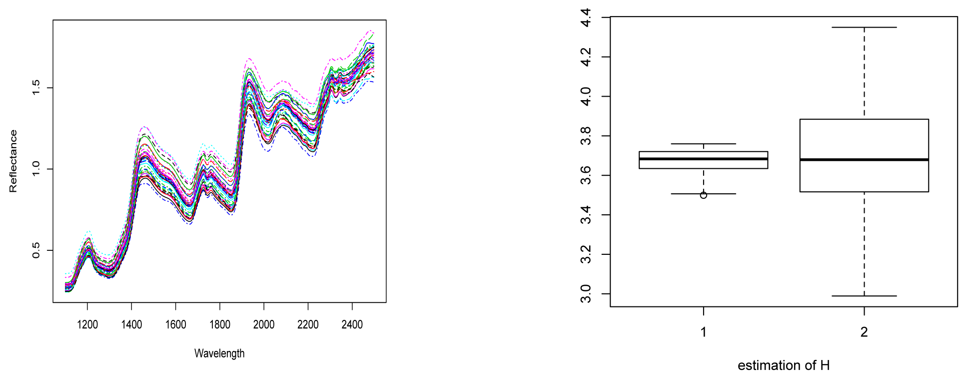
TABLE 3.8. Estimates in the roller height example

| $m$ | $\widehat{H}_n^{(1)}$ | $\widehat{\alpha}_{\text{OLS}}^{(0)}$ | $\widehat{H}_n^{(2)}$ | $\widehat{\alpha}_{\text{OLS}}^{(1)}$ |
|---|---|---|---|---|
| 2 | 0.63 | 0.63 | 0.77 | 0.77 |
| 4 | 0.50 | 0.51 | 0.63 | 0.65 |
| 6 | 0.38 | 0.39 | 0.49 | 0.51 |
| 8 | 0.35 | 0.33 | 0.44 | 0.42 |
| 10 | 0.30 | 0.28 | 0.39 | 0.35 |

regularity $r_0$ was supposed to be null. Indeed, our estimator $\widehat{r_0}$, directly used on the data with $b_n = 1/log(n)$, gives $\widehat{r_0} = 0$, with a value of $n^{4-2}\overline{\left(\Delta_2^{(1)} X\right)^2} = 1172345$. Next, we compute the values obtained for the estimation of $H$ by setting $(u, v) = (1, m)$ in our estimators, with $m$ in $\{2, 4, 6, 8, 10\}$. In Table 3.8, values of estimates proposed by Constantine and Hall (1994); Kent and Wood (1995, 1997) are also reported for comparison. These estimates are obtained by considering ordinary least squares estimators $\widehat{\alpha}_{\text{OLS}}^{(p)}$ (with $p$ adapted to the regularity of the process, supposed to be known in their work). Note that, with Kent and Wood's notation, we have in this case $H = 2\beta_0 = \alpha$ and for $(u, v) = (1, 2)$, one gets $\widehat{H}_n^{(1)} = \widehat{\alpha}_{\text{OLS}}^{(0)}$ and $\widehat{H}_n^{(2)} = \widehat{\alpha}_{\text{OLS}}^{(1)}$ but $\widehat{H}$ and $\widehat{\alpha}$ differ for other values of $v$. It should be observed that our estimators present a similar sensitivity to the choice of $m$, with perhaps a slightly greater robustness of our estimators against high values of $m$.

TABLE 3.9. Estimates in the biscuit example

| | $m=2$ | $m=4$ | $m=6$ | $m=8$ | $m=10$ |
|---|---|---|---|---|---|
| $\widehat{H}_n^{(1)}$ | 3.60 (0.12) | 3.67 (0.07) | 3.65 (0.05) | 3.62 (0.04) | 3.59 (0.04) |
| $\widehat{\alpha}_{\text{OLS}}^{(1)}$ | 3.60 (0.12) | 3.67 (0.07) | 3.66 (0.05) | 3.63 (0.04) | 3.60 (0.03) |
| $\widehat{H}_n^{(2)}$ | 2.84 (0.45) | 3.69 (0.30) | 3.83 (0.24) | 3.84 (0.19) | 3.83 (0.16) |
| $\widehat{\alpha}_{\text{OLS}}^{(2)}$ | 2.84 (0.45) | 3.67 (0.31) | 3.91 (0.23) | 3.98 (0.18) | 3.99 (0.14) |




(a) (b)

FIGURE 3.4. (a) Curve drawing reflectance in function of wavelength, varying between 1100 and 2498. (b) Box-plots drawing the results for both estimators on the right $\widehat{H}_n^{(1)}$ on the left $\widehat{H}_n^{(2)}$ for the 39 curves and $(u, v) = (1, 4)$.

In order to compare the (empirical) variances of these estimators, we consider a second example introduced by Brown et al. (2001). The experiment involved varying the composition of biscuit dough pieces and data consist in near infrared reflectance (NIR) spectra for the same dough. The 40 curves are graphed on the figure 3.4. Each represents the near-infrared spectrum reflectance measure at each 2 nanometers from 1100 to 2498 nm, then 700 observation points for each biscuit. According to Brown et al. (2001), the observation 23 appears as an outlier. We estimate $r_0$ for each of the left 39 curves, using the threshold $b_n = 1$, which gives $\widehat{r_0} = 1$ for each curve and an averaged mean quadratic variation $\overline{n^{2r-2}\big(\Delta_r^{(1)}X\big)^2}$ equal to 0.33 when $r = 2$ and 122133 when $r = 3$, this explosion leads us to the choice $\widehat{r_0} = 3 - 2 = 1$. We turn to estimation of $H$, to compare our estimators with $\widehat{\alpha}_{\text{OLS}}^{(p)}$ where $p = 1$ corresponds to the choice $(1, -2, 1)$ for $a_{j,r}$ and $p = 2$ to the choice $(-1, 3, -3, 1)$. The results are summarised in Table 3.9 and it appears that, for order $\widehat{r_0} + 2 = 3$, our estimator $\widehat{H}_n^{(2)}$ seems to be less sensitive to the choice of $m$. To conclude this part, it should be noticed that for the 23rd curve, the choice $m = 4$ gives $\widehat{H}_n^{(1)} = 3.64$ and $\widehat{H}_n^{(2)} = 3.55$. It appears that, in both cases, these values belong to the interquartile range obtained from the 39 curves, so at least concerning the regularity, the curve 39 should not be considered as an outlier.

3.3. **Plug-in estimation : results and simulation.** A classical and interesting topic is approximation and/or integration of a sampled path. An extensive literature may be found on these topics with a detailed overview in the recent monograph of Ritter (2000).

The general framework is as follows : let $X = \{X_t,\, t \in [0,1]\}$, be observed at sampled times $t_{0,n}, \dots, t_{n,n}$ over [a,b], more simply denoted by $t_0, \dots, t_n$. Approximation of $X(\cdot)$ consists in interpolation of the path on $[a,b]$, while weighted integration is the calculus of $\mathcal{I}_\rho = \int_a^b X(t)\rho(t)\,\mathrm{d}t$ for some positive and continuous weight function $\rho$. These problems are closely linked, see e.g. Ritter (2000) p. 19-21. Closely to our framework of local stationary derivatives, we may refer more specifically to works of Plaskota et al. (2004) for approximation and Benhenni (1998) for integration. For sake of clarity, we give a brief summary of their obtained results. In the following, we denote by $\mathcal{H}(r_0, \beta_0)$ the family of Gaussian processes having $r_0$ derivatives in quadratic mean and $r_0$-th derivative with Hölderian regularity of order $\beta_0 \in ]0,1[$. For measurable $g_i(\cdot)$, we consider the approximation $\mathcal{A}_{n,g}(t) = \sum_{i=0}^n X(t_i)g_i(t)$ and $e_\rho(\mathcal{A}_{n,g}) = \Big(\int_a^b \mathbb{E}\,|X(t) - \mathcal{A}_{n,g}(t)|^2\,\rho(t)\,\mathrm{d}t\Big)^{1/2}$ will represent the weighted and integrated $L^2$-error. For $X \in \mathcal{H}(r_0, \beta_0)$ and known $(r_0, \beta_0)$, Plaskota et al. (2004) have shown that

$$0 < c(r_0, \beta_0) \le \varliminf_{n\to\infty} n^{r_0+\beta_0} \inf_g e_\rho(\mathcal{A}_{n,g}) \le \varlimsup_{n\to\infty} n^{r_0+\beta_0} \inf_g e_\rho(\mathcal{A}_{n,g}) \le C(r_0, \beta_0) < +\infty$$

for equidistant sampled times $t_1, \dots, t_n$ and Gaussian processes defined and observed on the half-line $[0, +\infty[$. Of course, optimal choices of functions $g_i$, giving a minimal error, depend on the unknown covariance function of $X$.

Concerning weighted integration, the quadrature is denoted by $\mathcal{Q}_{n,d} = \sum_{i=0}^n X(t_i)d_i$ with well-chosen constants $d_i$ (typically, one may take $d_i = \int_a^b g_i(t)\,\mathrm{d}t$). For known $(r_0, \beta_0)$, a short list of references could be:

- Sacks and Ylvisaker (1968, 1970) with $r_0 = 0$ or 1, $\beta_0 = \frac{1}{2}$ and known covariance,
- Benhenni and Cambanis (1992) for arbitrary $r_0$ and $\beta_0 = \frac{1}{2}$,
- Stein (1995) for stationary processes and $r_0 + \beta_0 < \frac{1}{2}$,
- Ritter (1996) for minimal error, under Sacks and Ylvisaker's conditions, and with arbitrary $r_0$.

Let us set $e_\rho(\mathcal{Q}_{n,d}) = \Big(\mathbb{E}\,|I_\rho - \mathcal{Q}_{n,d}|^2\Big)^{1/2}$, the mean square error of integration. In the stationary case and for known $r_0$, Benhenni (1998) established the following exact behaviour: If $\rho \in C^{r_0+3}([a,b])$ then for some given quadrature $\mathcal{Q}_{n,d^*(r_0)}$ on $[a,b]$,

$$n^{r_0+\beta_0+\frac{1}{2}}\, e_\rho\big(\mathcal{Q}_{n,d^*(r_0)}\big) \xrightarrow[n\to\infty]{} c_{r_0,\beta_0}\Big(\int_a^b \rho^2(t)\psi^{-(2(r_0+\beta_0)+1)}(t)\,\mathrm{d}t\Big)^{\frac{1}{2}}$$

where $\psi$ is the density relative to the regular sampling $\{t_1, \dots, t_n\}$. Moreover, following Ritter (1996), it appears that this last result is optimal under Sacks and Ylvisaker's conditions. Finally, Istas and Laredo (1997) have proposed a quadrature, requiring only an upper bound on $r_0$, with an error of order $\mathcal{O}\big(n^{-(r_0+\beta_0+\frac{1}{2})}\big)$.

All these results shown the importance of well estimating $r_0$ and motivate ourself to focus on plugged-in interpolators, namely those using Lagrange polynomial of order estimating by $\widehat{r_0}$. More precisely, for equidistant sampling $t_i = i\delta_n$, $i = 0, \dots, n$, $\delta_n \to 0$, $n\delta_n \to T > 0$,

Lagrange interpolation of order $r \geq 1$ is defined by

$$\widetilde{X}_r(t) = \sum_{i=0}^{r} L_{i,k,r}(t) X\big((kr+i)\delta_n\big), \text{ with } L_{i,k,r}(t) = \prod_{\substack{j=0 \\ j\neq i}}^{r} \frac{(t-(kr+j)\delta_n)}{(i-j)\delta_n}, \tag{3.1}$$

for $t \in \mathcal{I}_k := \Big[kr\delta_n, (k+1)r\delta_n\Big]$, $k = 0, \ldots, \lfloor \frac{n}{r} \rfloor - 2$ and $\mathcal{I}_{\lfloor \frac{n}{r} \rfloor - 1} = \Big[\lfloor (\frac{n}{r} \rfloor - 1) r\delta_n, T\Big]$.

Our plugged method will consist in the approximation given by $\mathcal{A}_{n,L}(t) = \widetilde{X}_{\max(\widehat{r_0},1)}(t)$, $t \in [0,T]$, and quadrature by $\mathcal{Q}_{n,L} = \int_0^T \widetilde{X}_{\widehat{r_0}+1}(t)\,\rho(t)\,\mathrm{d}t$. Indeed, Lagrange polynomials are of easy implementation and by the result of Plaskota et al. (2004), they reach the optimal rate of approximation, $n^{-(r_0+\beta_0)}$ without requiring knowledge of covariance. Also we will show that the associate quadrature has the expected rate $n^{-(r_0+\beta_0+\frac{1}{2})}$. For unknown $(r_0, \beta_0)$, a first result of plugged approximation is derived in Blanke and Vial (2011): if Assumption A2.1 holds, we have for $T = 1$, $\rho \equiv 1$ and some constant $D(r_0, \beta_0)$:

$$\lim_{n\to\infty} n^{2(r_0+\beta_0)} \int_0^1 \mathbb{E}\big(X(t) - \widetilde{X}_{\max(\widehat{r_0},1)}(t)\big)^2\,\mathrm{d}t = D(r_0, \beta_0).$$

In the weighted case and for $T > 0$, we obtain the following asymptotic bounds.

**Theorem 3.1.** *Suppose that Assumption A2.1 holds and consider a positive and continuous weight function $\rho$.*

(a) *Under condition A2.2(1), we have*

$$e_\rho(\mathrm{app}(\widehat{\mathrm{r}_0})) := \Big(\int_0^{\mathrm{T}} \mathbb{E}\left|\mathrm{X(t)} - \widetilde{\mathrm{X}}_{\max(\widehat{\mathrm{r}_0},1)}(\mathrm{t})\right|^2 \rho(\mathrm{t})\,\mathrm{dt}\Big)^{1/2} = \mathcal{O}\big(\delta_{\mathrm{n}}^{\mathrm{r}_0+\beta_0}\big),$$

(b) *if condition A2.2(2) holds:*

$$e_\rho(\mathrm{int}(\widehat{\mathrm{r}_0})) := \Big(\mathbb{E}\left|\int_0^{\mathrm{T}} \mathrm{X(t)}\rho(\mathrm{t})\,\mathrm{dt} - \int_0^{\mathrm{T}} \widetilde{\mathrm{X}}_{\widehat{\mathrm{r}_0}+1}(\mathrm{t})\rho(\mathrm{t})\,\mathrm{dt}\right|^2\Big)^{1/2} = \mathcal{O}\big(\delta_{\mathrm{n}}^{\mathrm{r}_0+\beta_0+\frac{1}{2}}\big).$$

In conclusion, expected rates for approximation and integration are reached by plugged Lagrange piecewise polynomials. Of course if $r_0$ is known, this last result holds true with $\widehat{r_0}$ replaced by $r_0$. The figure 3.3 is obtained using 1000 simulated sample paths observed in equally spaced points on $[0,1]$. This figure illustrates results of approximation for different processes. The logarithm of empirical integrated mean square error (in short IMSE), i.e. $e_1^2(\mathrm{app}(\widehat{\mathrm{r}_0}))$ is drawn in function of $\ln(n)$. We may notice that we obtain straight lines with slope very near to $-H = -2(r_0 + \beta_0)$. Since the Ornstein-Uhlenbeck process is a scaled time-transformed Wiener proces, intercepts are different contrary to stationary versus non stationary CARMA processes.

## 4. Proofs of theoretical results

4.1. **Proof of Proposition 2.1.** **A.** Let us begin with general expressions of $\mathbb{E}\Big(\Delta_{r,k}^{(u)} X \Delta_{r,\ell}^{(u)} X\Big)$ useful for the sequel.

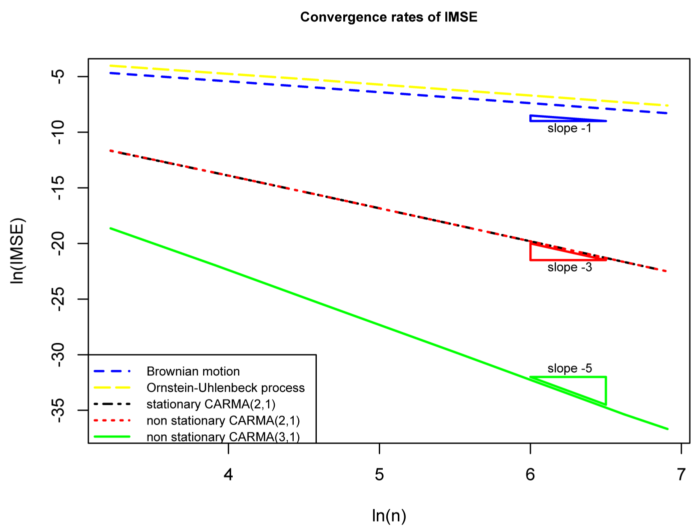



FIGURE 3.5. Logarithm of $e_1^2(\mathrm{app}(\widehat{\mathrm{r}_0}))$, i.e. the IMSE in function of $\ln(n)$, for different type of processes. The dashed line correspond to Brownian motion, long dashed line to O.U., dashed line to non stationary CARMA(2,1), dotted-dashed line to stationary CARMA(2,1) and solid line to non stationary CARMA(3,1). The small triangles near lines are here to indicate the theoretic slope.

First for $\mathbb{L}^{(p,p)}(s,t) = \mathbb{E}\big(X^{(p)}(s)X^{(p)}(t)\big)$ $(p \geq 0)$, the relation (2.1) is equivalent to

$$\lim_{h\to 0} \sup_{\substack{s,t\in[0,T] \\ |s-t|\leq h, s\neq t}} \left| \frac{\mathbb{L}^{(r_0,r_0)}(s,s) + \mathbb{L}^{(r_0,r_0)}(t,t) - 2\mathbb{L}^{(r_0,r_0)}(s,t)}{|s-t|^{2\beta_0}} - d_0(t) \right| = 0. \tag{4.1}$$

We set $\dot{k} = k\delta_n$, $\dot{k}_{iu} = (k+iu)\delta_n$, $\dot{k}_{iuv} = (k+iuv)\delta_n$, $\dot{\ell} = \ell\delta_n$, $\dot{\ell}_{ju} = (\ell+ju)\delta_n$ and $\dot{\ell}_{juw} = (\ell+juw)\delta_n$. Next, from the definition of $\Delta_{r,k}^{(u)} X$ given in (2.2), we get

$$\mathbb{E}(\Delta_{r,k}^{(u)} X \Delta_{r,\ell}^{(u)} X) = \sum_{i,j=0}^{\overline{r}} a_{i,r} a_{j,r} \mathbb{L}^{(0,0)}(\dot{k}_{iu}, \dot{\ell}_{ju}).$$

Case $r_0 = 0$. As $\sum_{i=0}^{\overline{r}} a_{i,r} = 0$, we have:

$$\mathbb{E}(\Delta_{r,k}^{(u)} X \Delta_{r,\ell}^{(u)} X) = \sum_{i=0}^{\overline{r}} \sum_{j=0}^{\overline{r}} a_{i,r} a_{j,r} \Big\{ \mathbb{L}^{(0,0)}(\dot{k}_{iu}, \dot{\ell}_{ju}) - \frac{1}{2}\mathbb{L}^{(0,0)}(\dot{k}_{iu}, \dot{k}_{iu}) - \frac{1}{2}\mathbb{L}^{(0,0)}(\dot{\ell}_{ju}, \dot{\ell}_{ju}) \Big\}. \tag{4.2}$$

Case $r_0 \geq 1$. We apply multiple Taylor series expansions with integral remainder. Next, the properties $\sum_{i=0}^{\overline{r}} a_{i,r} i^p = 0$ for $p = 0, \ldots, r-1$ (and convention $0^0 = 1$) induce :

$$\text{(4.3)} \quad \mathbb{E}(\Delta_{r,k}^{(u)} X \Delta_{r,\ell}^{(u)} X) = \sum_{i,j=0}^{\overline{r}} a_{i,r} a_{j,r} (u^2 ij\delta_n^2)^{r^*} \iint_{[0,1]^2} \frac{(1-v)^{r^*-1}(1-w)^{r^*-1}}{((r^*-1)!)^2} \mathbb{L}^{(r^*,r^*)}(\dot{k}_{iuv}, \dot{\ell}_{juw})\, \mathrm{d}v\mathrm{d}w$$

where we have set $r^* = \min(r_0, r) \geq 1$.

**B.** From expressions (4.2)-(4.3), we now derive the asymptotic behaviour of $\mathbb{E}\Big(\overline{(\Delta_r X^{(u)})^2}\Big)$. Case $r_0 \geq 1$, $r = r_0 + 1$ or $r = r_0 + 2$. In this case, $r^* = r_0 \leq r - 1$. From (4.3) and the property $\sum_{i=0}^{\overline{r}} a_{i,r} i^{r_0} = 0$, we may write

$$\text{(4.4)} \quad \mathbb{E}(\Delta_{r,k}^{(u)} X)^2 = \sum_{i,j=0}^{\overline{r}} a_{i,r} a_{j,r} (u^2 ij\delta_n^2)^{r_0} \iint_{[0,1]^2} \frac{(1-v)^{r_0-1}(1-w)^{r_0-1}}{((r_0-1)!)^2} \Big\{ \mathbb{L}^{(r_0,r_0)}(\dot{k}_{iuv}, \dot{k}_{juw}) - \frac{1}{2}\mathbb{L}^{(r_0,r_0)}(\dot{k}_{iuv}, \dot{k}_{iuv}) - \frac{1}{2}\mathbb{L}^{(r_0,r_0)}(\dot{k}_{juw}, \dot{k}_{juw}) \Big\}\, \mathrm{d}v\mathrm{d}w$$

Using the property $\left|\dot{k}_{iuv} - \dot{k}_{juw}\right| = |iv - jw|\, u\delta_n$, we decompose $\mathbb{E}(\Delta_{r,k}^{(u)} X)^2$ into $S_{1n}(k) + S_{2n}(k) + S_{3n}(k)$ where

$$S_{1n}(k) = -\sum_{i,j=0}^{\overline{r}} a_{i,r} a_{j,r} (ij)^{r_0} (u\delta_n)^H \iint_{[0,1]^2} \frac{((1-v)(1-w))^{r_0-1}}{2((r_0-1)!)^2} |iv - jw|^{2\beta_0}\, d_0(\dot{k})\, \mathrm{d}v\mathrm{d}w$$

$$S_{2n}(k) = -\sum_{i,j=0}^{\overline{r}} a_{i,r} a_{j,r} (ij)^{r_0} (u\delta_n)^H \iint_{[0,1]^2} \frac{((1-v)(1-w))^{r_0-1}}{2((r_0-1)!)^2} |iv - jw|^{2\beta_0} \Big\{ d_0(\dot{k}_{juw}) - d_0(\dot{k}) \Big\}\, \mathrm{d}v\mathrm{d}w$$

and $S_{3n}(k)$ is defined by

$$-\sum_{i,j=0}^{\overline{r}} a_{i,r} a_{j,r} (ij)^{r_0} (u\delta_n)^H \iint_{[0,1]^2} \frac{((1-v)(1-w))^{r_0-1}}{2((r_0-1)!)^2} |iv - jw|^{2\beta_0} \Bigg[ -d_0(\dot{k}_{juw}) + \Bigg\{ \frac{\mathbb{L}^{(r_0,r_0)}(\dot{k}_{iuv}, \dot{k}_{iuv}) + \mathbb{L}^{(r_0,r_0)}(\dot{k}_{juw}, \dot{k}_{juw}) - 2\mathbb{L}^{(r_0,r_0)}(\dot{k}_{iuv}, \dot{k}_{juw})}{|(iv - jw)u\delta_n|^{2\beta_0}} \Bigg\} \Bigg]\, \mathrm{d}v\mathrm{d}w.$$

From the locally stationary condition (4.1), we get that $\displaystyle\frac{\delta_n^{-2(r_0+\beta_0)}}{n_r + 1} \sum_{k=0}^{n_r} S_{3n}(k) \xrightarrow[n\to\infty]{} 0$. Concerning $S_{2n}(k)$, uniform continuity and boundedness of the function $d_0$ on $[0, T]$ implies

$$\frac{\delta_n^{-2(r_0+\beta_0)}}{n_r + 1} \sum_{k=0}^{n_r} S_{2n}(k) \leq C(\overline{r}, r_0) \sum_{j=0}^{\overline{r}} \sup_{w\in[0,1]} \sup_{t\in[0,(n_r)\delta_n]} |d_0(t + juw\delta_n) - d_0(t)| \xrightarrow[n\to\infty]{} 0.$$

Finally, since $n\delta_n \xrightarrow[n\to\infty]{} T$:

$$\frac{\delta_n^{-2(r_0+\beta_0)}}{n_r+1}\sum_{k=0}^{n_r} S_{1n}(k) \xrightarrow[n\to\infty]{} u^H \ell(r,r_0,\beta_0)$$

with

$$\ell(r,r_0,\beta_0) = \\ -\sum_{i,j=0}^{\overline{r}} \frac{a_{i,r}a_{j,r}(ij)^{r_0}}{2((r_0-1)!)^2}\Big(\int_{[0,1]^2}((1-v)(1-w))^{r_0-1}\,|iv-jw|^{2\beta_0}\,\mathrm{d}v\mathrm{d}w\Big)\times\Big(\frac{1}{T}\int_0^T d_0(t)\,\mathrm{d}t\Big).$$

Next by performing elementary but tedious multiple integrations by part and using the property $\sum_{j=1}^{\overline{r}} a_{j,r}j^p = 0$ for $p=1,\dots,r-1$, we arrive at the following simpler form of $\ell(r,r_0,\beta_0)$ given in (2.3):

$$\ell(r,r_0,\beta_0) = \frac{(-1)^{r_0+1}\int_0^T d_0(t)\,\mathrm{d}t}{2T(2\beta_0+2r_0)\cdots(2\beta_0+1)}\sum_{i,j=0}^{\overline{r}} a_{i,r}a_{j,r}\,|i-j|^{2r_0+2\beta_0}\,.$$

Case $r_0=0$, $r=r_0+1$ or $r=r_0+2$. The proof is the same but starting from (4.2) (with $\ell=k$) rather than the relation (4.3).
Case $r_0\geq 1$ and $r=1,\dots,r_0$. In this case, $r^*=r$ and starting again from the relation (4.3), one gets

$$\mathbb{E}(\Delta_{r,k}^{(u)}X)^2 = u^{2r}\sum_{i,j=0}^{\overline{r}} a_{i,r}a_{j,r}\big(ij\delta_n^2\big)^r\iint_{[0,1]^2}\frac{(1-v)^{r-1}(1-w)^{r-1}}{((r-1)!)^2}\mathbb{L}^{(r,r)}(\dot{k}_{iuv},\dot{k}_{juw})\,\mathrm{d}v\mathrm{d}w.$$

The result follows from uniform continuity of $\mathbb{L}^{(r,r)}$, $r=1,\dots,r_0$ and the property $\mathbb{E}\big(X^{(r)}(t)\big)^2 = \mathbb{L}^{(r,r)}(t,t)$.

4.2. **Auxiliary results.** The following lemmas give some useful results on the asymptotic behaviour of $\mathbb{C}_r(k,\ell)$ and $\mathbb{C}_r^2(k,\ell)$ with $\mathbb{C}_r(k,\ell)=\mathrm{Cov}\big(\Delta_{r,k}^{(u)}X,\Delta_{r,\ell}^{(u)}X\big)$.

**Lemma 4.1.** *Suppose that Assumption A2.1 is fulfilled and let $u$ be a positive integer.*
*(i) Under Assumption A2.2(1) and for $r=r_0+1$ or $r=r_0+2$, one obtains*

$$\max_{k=0,\dots,n_r}\sum_{\ell=0}^{n_r}|\mathbb{C}_r(k,\ell)| = \begin{cases}\mathcal{O}\big(n^{-H}\big) & \text{if } 0<\beta_0<\frac12,\\ \mathcal{O}\big(n^{-H}\ln n\big) & \text{if } \beta_0=\frac12,\\ \mathcal{O}\big(n^{-(2r_0+1)}\big) & \text{if } \frac12<\beta_0<1\ ;\end{cases}$$

*and*

$$\sum_{k=0}^{n_r}\sum_{\ell=0}^{n_r}\mathbb{C}_r^2(k,\ell) = \begin{cases}\mathcal{O}\big(n^{-2H+1}\big) & \text{if } 0<\beta_0<\frac34,\\ \mathcal{O}\big(n^{-2H+1}\ln n\big) & \text{if } \beta_0=\frac34,\\ \mathcal{O}\big(n^{-(4r_0+2)}\big) & \text{if } \frac34<\beta_0<1.\end{cases}$$

*(ii) Under Assumption A2.2(2) and for $r = r_0 + 2$, one obtains* $\max_{k=0,\dots,n_r} \sum_{\ell=0}^{n_r} |\mathbb{C}_r(k,\ell)| = \mathcal{O}(n^{-H})$ *and* $\sum_{k=0}^{n_r}\sum_{\ell=0}^{n_r} \mathbb{C}_r^2(k,\ell) = \mathcal{O}(n^{-2H+1})$.

*(iii) If* $r = 1,\dots,r_0$ *(with* $r_0 \geq 1$*), then* $\max_{k=0,\dots,n_r} \sum_{\ell=0}^{n_r} |\mathbb{C}_r(k,\ell)| = \mathcal{O}\Big(n^{-2r+1}\Big)$ *and* $\sum_{k=0}^{n_r}\sum_{\ell=0}^{n_r} \mathbb{C}_r^2(k,\ell) = \mathcal{O}\Big(n^{-4r+2}\Big)$.

*Proof.* Setting $\mu(t) = \mathbb{E}\big(X(t)\big)$, one has

$$\mathbb{C}_r(k,\ell) = \sum_{i,j=0}^{\overline{r}} a_{i,r}a_{j,r}\big\{\mathbb{L}^{(0,0)}(\dot{k}_{iu},\dot{\ell}_{ju}) - \mu(\dot{k}_{iu})\mu(\dot{\ell}_{ju})\big\}.$$

Similarly to (4.2)-(4.3), we get the expansion

$$\mathbb{C}_r(k,\ell) = \sum_{i,j=0}^{\overline{r}} a_{i,r}a_{j,r}\big(u^2 ij\delta_n^2\big)^{r^*}\iint_{[0,1]^2} \frac{(1-v)^{r^*-1}(1-w)^{r^*-1}}{((r^*-1)!)^2}\mathbb{K}^{(r^*,r^*)}(\dot{k}_{iuv},\dot{\ell}_{juw})\,\mathrm{d}v\mathrm{d}w \tag{4.5}$$

for $r^* = \min(r_0, r) \geq 1$ while if $r_0 = 0$:

$$\mathbb{C}_r(k,\ell) = \sum_{i=0}^{\overline{r}}\sum_{j=0}^{\overline{r}} a_{i,r}a_{j,r}\mathbb{K}(\dot{k}_{iu},\dot{\ell}_{ju}). \tag{4.6}$$

(i) Case $r_0 \geq 1$, $r = r_0 + 1$ or $r = r_0 + 2$ and A2.2(1) satisfied. In this case $r^* = r_0$ and we have the bound:

$$\max_{k=0,\dots,n_r} \sum_{\ell=0}^{n_r} |\mathbb{C}_r(k,\ell)| \leq U_{1n}(\overline{r}, r_0) + U_{2n}(\overline{r}, r_0) + U_{3n}(\overline{r}, r_0)$$

with $U_{1n}(\overline{r}, r_0) = \max_{k=u\overline{r}+1,\dots,n_r} \sum_{\ell=0}^{k-u\overline{r}-1} |\mathbb{C}_r(k,\ell)|$, $U_{2n}(\overline{r}, r_0) = \max_{k=0,\dots,n-2u\overline{r}-1} \sum_{\ell=k+u\overline{r}+1}^{n_r} |\mathbb{C}_r(k,\ell)|$ and $U_{3n}(\overline{r}, r_0) = \max_{k=0,\dots,n_r} \sum_{\ell=\max(0,k-u\overline{r})}^{\min(n_r,k+u\overline{r})} |\mathbb{C}_r(k,\ell)|$.

First, consider the sum $U_{1n}(\overline{r}, r_0) + U_{2n}(\overline{r}, r_0)$ where $|k-\ell| \geq u\overline{r}+1$. We have $\sum_{i=0}^{\overline{r}} a_{i,r} i^{r_0} = 0$ for $r = r_0 + 1$ or $r = r_0 + 2$ and (4.5) may be written as follows:

$$\begin{aligned}\mathbb{C}_r(k,\ell) = \sum_{i,j=0}^{\overline{r}} a_{i,r}a_{j,r}\big(u^2 ij\delta_n^2\big)^{r_0}\iint_{[0,1]^2} \frac{(1-v)^{r_0-1}(1-w)^{r_0-1}}{((r_0-1)!)^2}\Big\{&\mathbb{K}^{(r_0,r_0)}(\dot{k}_{iuv},\dot{\ell}_{juw}) \\ -\mathbb{K}^{(r_0,r_0)}(\dot{k},\dot{\ell}_{juw}) - \mathbb{K}^{(r_0,r_0)}(\dot{k}_{iuv},\dot{\ell}) + &\mathbb{K}^{(r_0,r_0)}(\dot{k},\dot{\ell})\Big\}\,\mathrm{d}v\mathrm{d}w\end{aligned} \tag{4.7}$$

and since $[\dot{k}, \dot{k}_{iuv}]$ and $[\dot{\ell}, \dot{\ell}_{juw}]$ are distinct for $|k-l| \geq u\overline{r}+1$:

$$\mathbb{C}_r(k,\ell) = \sum_{i,j=0}^{\overline{r}} a_{i,r}a_{j,r}\big(u^2 ij\delta_n^2\big)^{r_0} \iint_{[0,1]^2} \frac{(1-v)^{r_0-1}(1-w)^{r_0-1}}{((r_0-1)!)^2} \times \int_{\dot{k}}^{\dot{k}_{iuv}}\int_{\dot{\ell}}^{\dot{\ell}_{juw}} \mathbb{K}^{(r_0+1,r_0+1)}(s,t)\,\mathrm{d}s\mathrm{d}t\mathrm{d}v\mathrm{d}w.$$

By this way, the condition A2.2(1) gives the bound

$$|U_{1n}(\overline{r},r_0)+U_{2n}(\overline{r},r_0)| \leq C_1(\overline{r},r_0)\delta_n^{2r_0} \times \Bigg(\max_{k=u\overline{r}+1,\dots,n_r} \sum_{\ell=0}^{k-u\overline{r}-1} \int_{\dot{k}}^{\dot{k}+u\overline{r}\delta_n}\int_{\dot{\ell}}^{\dot{\ell}+u\overline{r}\delta_n} (t-s)^{-2(1-\beta_0)}\mathrm{d}s\mathrm{d}t + \max_{k=0,\dots,n-2u\overline{r}-1} \sum_{\ell=k+u\overline{r}+1}^{n_r} \int_{\dot{k}}^{\dot{k}+u\overline{r}\delta_n}\int_{\dot{\ell}}^{\dot{\ell}+u\overline{r}\delta_n} (s-t)^{-2(1-\beta_0)}\mathrm{d}s\mathrm{d}t\Bigg)$$

involving, in turn, that

$$|U_{1n}(\overline{r},r_0)+U_{2n}(\overline{r},r_0)| \leq C_2(\overline{r},r_0)n^{-H}\sum_{i=1}^{n} i^{-2(1-\beta_0)}$$

which is of order $n^{-H}$ if $0<\beta_0<\frac{1}{2}$, $n^{-H}\ln n$ if $\beta_0=\frac{1}{2}$ and $n^{-2r_0-1}$ if $\beta_0>\frac{1}{2}$.

Next, for $U_{3n}(\overline{r},r_0)$ where $|k-\ell|\leq u\overline{r}$, we proceed similarly as in the proof of Proposition 2.1, on basis of the following decomposition derived from (4.5):

$$\begin{aligned}\mathbb{C}_r(k,\ell) = &-\frac{1}{2}\sum_{i,j=0}^{\overline{r}} a_{i,r}a_{j,r}\iint_{[0,1]^2}\big(u^2ij\delta_n^2\big)^{r_0}\frac{\big((1-v)(1-w)\big)^{r_0-1}}{((r_0-1)!)^2}\left|\dot{k}_{iuv}-\dot{\ell}_{juw}\right|^{2\beta_0}\\ &\left\{\frac{\mathbb{L}^{(r_0,r_0)}(\dot{k}_{iuv},\dot{k}_{iuv})+\mathbb{L}^{(r_0,r_0)}(\dot{\ell}_{juw},\dot{\ell}_{juw})-2\mathbb{L}^{(r_0,r_0)}(\dot{k}_{iuv},\dot{\ell}_{juw})}{\left|\dot{k}_{iuv}-\dot{\ell}_{juw}\right|^{2\beta_0}}-d_0(\dot{\ell}_{juw})\right\}\mathrm{d}v\mathrm{d}w\\ &-\frac{1}{2}\sum_{i,j=0}^{\overline{r}} a_{i,r}a_{j,r}\iint_{[0,1]^2}\big(u^2ij\delta_n^2\big)^{r_0}\frac{\big((1-v)(1-w)\big)^{r_0-1}}{((r_0-1)!)^2}\left|\dot{k}_{iuv}-\dot{\ell}_{juw}\right|^{2\beta_0}d_0(\dot{\ell}_{juw})\,\mathrm{d}v\mathrm{d}w\\ &-\Big(\sum_{i=0}^{\overline{r}} a_{i,r}\int_0^1 (ui\delta_n)^{r_0}\frac{(1-v)^{r_0-1}}{(r_0-1)!}\big(\mu^{(r_0)}(\dot{k}_{iuv})-\mu^{(r_0)}(\dot{k})\big)\,\mathrm{d}v\Big)\\ &\times\Big(\sum_{j=0}^{\overline{r}} a_{j,r}\int_0^1 (uj\delta_n)^{r_0}\frac{(1-w)^{r_0-1}}{(r_0-1)!}\big(\mu^{(r_0)}(\dot{\ell}_{juw})-\mu^{(r_0)}(\dot{\ell})\big)\,\mathrm{d}w\Big)\end{aligned}$$

The relation (2.1) gives a $o(\delta_n^H)$ for the first term while one obtains a $\mathcal{O}(\delta_n^H)$ for the following ones, using uniform continuity of $d_0$ and Cauchy-Schwarz inequality. The result is $U_{3n}(\overline{r},r_0)=\mathcal{O}\big(n^{-H}\big)$ as $n\delta_n\to T>0$.

Concerning $\sum_{k=0}^{n_r}\sum_{\ell=0}^{n_r}\mathbb{C}_r^2(k,\ell)$, the proof is essentially the same and we outline only the main steps. We start with $\sum_{k=0}^{n_r}\sum_{\ell=0}^{n_r}\mathbb{C}_r^2(k,\ell)\le V_{1n}(\overline{r},r_0)+V_{2n}(\overline{r},r_0)+V_{3n}(\overline{r},r_0)$ where $V_{1n}(\overline{r},r_0)=\sum_{k=u\overline{r}+1}^{n_r}\sum_{\ell=0}^{k-u\overline{r}-1}\mathbb{C}_r^2(k,\ell)$, $V_{2n}(\overline{r},r_0)=\sum_{k=0}^{n-2u\overline{r}-1}\sum_{\ell=k+u\overline{r}+1}^{n_r}\mathbb{C}_r^2(k,\ell)$ and $V_{3n}(\overline{r},r_0)=\sum_{k=0}^{n_r}\sum_{\ell=\max(0,k-u\overline{r})}^{\min(n_r,k+u\overline{r})}\mathbb{C}_r^2(k,\ell)$. From (4.7) and $r=r_0+1$, Assumption A2.2(1) gives

$$\begin{aligned}V_{1n}(\overline{r},r_0)+V_{2n}(\overline{r},r_0)\le C_1'(\overline{r},r_0)\delta_n^{4r_0}\times&\\ \Big(\sum_{k=u\overline{r}+1}^{n_r}\sum_{\ell=0}^{k-u\overline{r}-1}&\Big(\int_{\dot{k}}^{\dot{k}+u\overline{r}\delta_n}\int_{\dot{\ell}}^{\dot{\ell}+u\overline{r}\delta_n}(t-s)^{-2(1-\beta_0)}\mathrm{d}s\mathrm{d}t\Big)^2\\ +\sum_{k=0}^{n-2u\overline{r}-1}\sum_{\ell=k+u\overline{r}+1}^{n_r}&\Big(\int_{\dot{k}}^{\dot{k}+u\overline{r}\delta_n}\int_{\dot{\ell}}^{\dot{\ell}+u\overline{r}\delta_n}(s-t)^{-2(1-\beta_0)}\mathrm{d}s\mathrm{d}t\Big)^2\Big)\end{aligned}$$

implying in turn that $V_{1n}(\overline{r},r_0)+V_{2n}(\overline{r},r_0)\le C_2'(\overline{r},r_0)n^{-2H+1}\sum_{i=1}^{n}i^{-4(1-\beta_0)}$ which is of order $n^{-2H+1}$ if $0<\beta_0<\frac{3}{4}$, $n^{-2H+1}\ln n$ if $\beta_0=\frac{3}{4}$ and $n^{-4r_0-2}$ if $\beta_0>\frac{3}{4}$. Next, from (4.5), one may deduce that $V_{3n}(\overline{r},r_0)=\mathcal{O}(n^{-2H+1})$.

(i, followed) Case $r_0=0$, $r=r_0+1$ or $r=r_0+2$ and A2.2(1) satisfied. Both results are obtained by starting from (4.6) rather than (4.5).

(ii) Case where $r_0\ge 1$, $r=r_0+2$ and A2.2(2) holds. The single difference occurs for terms $U_{1n}$ (resp. $V_{1n}$) and $U_{2n}$ (resp. $V_{2n}$). Since $r=r_0+2$ and $\sum_{i=1}^{\overline{r}}i^{r_0+1}a_{i,r_0+2}=0$, one may add or subtract quantities as $iuv\delta_n\,\mathbb{K}^{(r_0+1,r_0)}(\dot{k},\dot{\ell}_{juw})$, $iuv\delta_n\,\mathbb{K}^{(r_0+1,r_0)}(\dot{k},\dot{\ell})$ or $iju^2vw\delta_n^2\,\mathbb{K}^{(r_0+1,r_0+1)}(\dot{k},\dot{\ell})$ without changing the result of (4.7). Thus,

$$\begin{aligned}\mathbb{C}_r(k,\ell)=\sum_{i,j=0}^{\overline{r}}a_{i,r_0+2}a_{j,r_0+2}\big(u^2ij\delta_n^2\big)^{r_0}\iint_{[0,1]^2}&\frac{(1-v)^{r_0-1}(1-w)^{r_0-1}}{((r_0-1)!)^2}\\ &\times\int_{\dot{k}}^{\dot{k}_{iuv}}\int_{\dot{\ell}}^{\dot{\ell}_{juw}}\iint_{\dot{k}}^{t}\!\!\int_{\dot{\ell}}^{s}\mathbb{K}^{(r_0+2,r_0+2)}(y,z)\ \mathrm{d}y\mathrm{d}z\mathrm{d}s\mathrm{d}t\mathrm{d}v\mathrm{d}w.\end{aligned}$$

By this way, the condition A2.2(2) gives the bound

$$|U_{1n}(\overline{r},r_0)+U_{2n}(\overline{r},r_0)| \le C_3(\overline{r},r_0)\delta_n^{2r_0}\times$$
$$\Big(\max_{k=u\overline{r}+1,\dots,n_r}\sum_{\ell=0}^{k-u\overline{r}-1}\int_{\dot{k}}^{\dot{k}+u\overline{r}\delta_n}\int_{\dot{\ell}}^{\dot{\ell}+u\overline{r}\delta_n}\iint_{\dot{k}\ \dot{\ell}}^{t\ \ s}(z-y)^{-2(2-\beta_0)}\mathrm{d}y\mathrm{d}z\mathrm{d}s\mathrm{d}t$$
$$+\max_{k=0,\dots,n-2u\overline{r}-1}\sum_{\ell=k+u\overline{r}+1}^{n_r}\int_{\dot{k}}^{\dot{k}+u\overline{r}\delta_n}\int_{\dot{\ell}}^{\dot{\ell}+u\overline{r}\delta_n}\iint_{\dot{k}\ \dot{\ell}}^{t\ \ s}(y-z)^{-2(2-\beta_0)}\mathrm{d}y\,\mathrm{d}z\mathrm{d}s\mathrm{d}t\Big)$$

so that $|U_{1n}(\overline{r},r_0)+U_{2n}(\overline{r},r_0)|\le C_4(\overline{r},r_0)n^{-H}\sum_{i=1}^{n}i^{-2(2-\beta_0)}=\mathcal{O}\big(n^{-H}\big)$, for all $\beta_0\in]0,1[$. On the other hand, one obtains that

$$|V_{1n}(\overline{r},r_0)+V_{2n}(\overline{r},r_0)|\le C_4'(\overline{r},r_0)n^{-2H+1}\sum_{i=1}^{n}i^{-4(2-\beta_0)}=\mathcal{O}\big(n^{-2H+1}\big).$$

(ii, followed) Case $r_0=0$, $r=r_0+2$ and A2.2(2) satisfied. The same methodology applies starting from (4.6) rather than (4.5).
(iii) Case $r=1,\dots,r_0$ (with $r_0\ge 1$). The expansion (4.5) becomes

$$\mathbb{C}_r(k,\ell)=\sum_{i,j=0}^{\overline{r}}a_{ir}a_{jr}\iint_{[0,1]^2}\big(iju^2\delta_n^2\big)^r\frac{((1-v)(1-w))^{r-1}}{((r-1)!)^2}\mathbb{K}^{(r,r)}(\dot{k}_{iuv},\dot{\ell}_{juw})\,\mathrm{d}v\mathrm{d}w$$

and both results follow from uniform continuity of $\mathbb{K}^{(r,r)}$ since $r\le r_0$.

□

The next proposition gives general exponential bounds, involved in the proofs of our main results.

**Proposition 4.1.** *Suppose that Assumption A2.1 is fulfilled, let $\eta_n(r)$ be some given positive sequence and $u\in\mathbb{N}^*$.*

*(i) For $r=r_0+1$ or $r=r_0+2$ and if Assumption A2.2(1) is satisfied, one obtains for*

$$\psi_n(\beta_0)=n^H\mathbb{1}_{]0,\frac12[}(\beta_0)+n^H(\ln n)\mathbb{1}_{\{\frac12\}}(\beta_0)+n^{H+2\beta_0-1}\mathbb{1}_{]\frac12,\frac34[}(\beta_0)$$
$$+n^{H+2\beta_0-1}(\ln n)^{-1}\mathbb{1}_{\{\frac34\}}(\beta_0)+n^{2r_0+2}\mathbb{1}_{]\frac34,1[}(\beta_0),\tag{4.8}$$

$$\varphi_{1n}(\beta_0)=n^{H+1}\mathbb{1}_{]0,\frac12[}(\beta_0)+n^{H+1}(\ln n)^{-1}\mathbb{1}_{\{\frac12\}}(\beta_0)+n^{2(r_0+1)}\mathbb{1}_{]\frac12,1[}(\beta_0),\tag{4.9}$$

$$\varphi_{2n}(\beta_0)=n^{2H+1}\mathbb{1}_{]0,\frac34[}(\beta_0)+n^{2H+1}(\ln n)^{-1}\mathbb{1}_{\{\frac34\}}(\beta_0)+n^{4(r_0+1)}\mathbb{1}_{]\frac34,1[}(\beta_0)\tag{4.10}$$

*and the conditions $\psi_n(\beta_0)\eta_n(r)\to 0$,*

$$\mathbb{P}\Big(\Big|\overline{(\Delta_r^{(u)}X)^2}-\mathbb{E}\Big(\overline{(\Delta_r^{(u)}X)^2}\Big)\Big|\ge\eta_n(r)\Big)=\mathcal{O}\Big(\exp\Big(-D_1(r)\,\varphi_{2n}(\beta_0)\eta_n^2(r)\Big)\Big)\tag{4.11}$$

*while for $\eta_n(r)$ such that $\psi_n(\beta_0)\eta_n(r)\to\zeta\in]0,+\infty]$:*

$$\mathbb{P}\Big(\Big|\overline{(\Delta_r^{(u)}X)^2}-\mathbb{E}\Big(\overline{(\Delta_r^{(u)}X)^2}\Big)\Big|\ge\eta_n(r)\Big)=\mathcal{O}\Big(\exp\Big(-D_2(r)\,\varphi_{1n}(\beta_0)\eta_n(r)\Big)\Big).\tag{4.12}$$

*(ii) For $r = r_0 + 2$ and if Assumption A2.2(2) holds, the condition $n^H \eta_n(r) \to 0$ implies*

$$\mathbb{P}\Big( \Big| \overline{(\Delta_r^{(u)} X)^2} - \mathbb{E}\Big( \overline{(\Delta_r^{(u)} X)^2} \Big) \Big| \geq \eta_n(r) \Big) = \mathcal{O}\Big( \exp\big( - D_3(r)\, n^{2H+1} \eta_n^2(r) \big) \Big) \tag{4.13}$$

*while if $n^H \eta_n(r) \to \zeta \in ]0, \infty]$:*

$$\mathbb{P}\Big( \Big| \overline{(\Delta_r^{(u)} X)^2} - \mathbb{E}\Big( \overline{(\Delta_r^{(u)} X)^2} \Big) \Big| \geq \eta_n(r) \Big) = \mathcal{O}\Big( \exp\big( - D_4(r)\, n^{H+1} \eta_n(r) \big) \Big). \tag{4.14}$$

*(iii) If $r = 1, \ldots, r_0$ (with $r_0 \geq 1$) and $n^{2r} \eta_n(r) \to \infty$:*

$$\mathbb{P}\Big( \Big| \overline{(\Delta_r^{(u)} X)^2} - \mathbb{E}\Big( \overline{(\Delta_r^{(u)} X)^2} \Big) \Big| \geq \eta_n(r) \Big) = \mathcal{O}\Big( \exp\big( - D_5(r) n^{2r} \eta_n(r) \big) \Big) \tag{4.15}$$

*where $D_i(r)$'s are positive constants not depending on $n$ and $\eta_n(r)$.*

*Proof.* For all $r \geq 1$, we may write

$$\mathbb{P}\Big( \Big| \overline{(\Delta_r^{(u)} X)^2} - \mathbb{E}\Big( \overline{(\Delta_r^{(u)} X)^2} \Big) \Big| \geq \eta_n(r) \Big) \leq S_1 + S_2 \tag{4.16}$$

with $S_1 = \mathbb{P}\Big( \Big| \sum_{k=0}^{n_r} (\Delta_{r,k}^{(u)} X - \mathbb{E}(\Delta_{r,k}^{(u)} X))^2 - \operatorname{Var}(\Delta_{r,k}^{(u)} X) \Big| > \frac{(n_r+1)\eta_n(r)}{2} \Big)$ and

$$S_2 = \mathbb{P}\Big( \Big| \sum_{k=0}^{n_r} (\mathbb{E}(\Delta_{r,k}^{(u)} X)) \big( \Delta_{r,k}^{(u)} X - \mathbb{E}(\Delta_{r,k}^{(u)} X) \big) \Big| > \frac{(n_r+1)\eta_n(r)}{4} \Big).$$

First, let $\{Y_i\}_{i=1,\ldots,d_n}$ be an orthonormal basis for the linear span of $\{\Delta_{r,k}^{(u)} X\}_{k=0,\ldots,n_r}$ (so that $Y_i$ are i.i.d. with density $\mathcal{N}(0,1)$). We can write

$$\Delta_{r,k}^{(u)} X - \mathbb{E}\big(\Delta_{r,k}^{(u)} X\big) = \sum_{i=1}^{d_n} \operatorname{Cov}\big(\Delta_{r,k}^{(u)} X, Y_i\big) Y_i := \sum_{i=1}^{d_n} b_{k,i} Y_i\,.$$

Next, if $Y = (Y_1, \ldots, Y_{d_n})^\top$, we obtain

$$\sum_{k=0}^{n_r} (\Delta_{r,k}^{(u)} X - \mathbb{E}\Delta_{r,k}^{(u)} X)^2 = \sum_{i,j=1}^{d_n} c_{i,j} Y_i Y_j = Y^\top C Y \quad \text{and} \quad \sum_{k=0}^{n_r} \operatorname{Var}(\Delta_{r,k}^{(u)} X) = \sum_{i}^{d_n} c_{i,i}$$

with $c_{i,j} = \sum_{k=0}^{n_r+1} b_{ki} b_{kj}$ so that for $C = \big(c_{i,j}\big)_{\substack{i=1,\ldots,d_n\\ j=1,\ldots,d_n}}$ and $B = \big(b_{k,j}\big)_{\substack{k=0,\ldots,n_r+1\\ j=1,\ldots,d_n}}$, one gets $C = B^\top B$ where $C$ is a real, symmetric and positive semidefinite matrix. There exists an orthogonal matrix $P$ such that $\operatorname{diag}(\lambda_1, \ldots, \lambda_{d_n}) = P^\top C P$, for $\lambda_i$ eigenvalues of $C$. Then we can transform the quadratic form as: $\sum_{k=0}^{n_r} (\Delta_{r,k}^{(u)} X - \mathbb{E}(\Delta_{r,k}^{(u)} X))^2 = \sum_{i=1}^{d_n} \lambda_i (P^\top Y)_i^2$ where $(P^\top Y)_i$ denotes the $i$-th component of the $(d_n \times 1)$ vector $P^\top Y$. Since $\sum_{i=1}^{d_n} c_{ii} = \sum_{i=1}^{d_n} \lambda_i = \sum_{k=0}^{n_r} \operatorname{Var}(\Delta_{r,k}^{(u)} X)$, we arrive at $S_1 = \mathbb{P}\Big( \Big| \sum_{i=1}^{d_n} \lambda_i \big( (P^\top Y)_i^2 - 1 \big) \Big| \geq \frac{(n_r+1)\eta_n(r)}{2} \Big)$. Now, with

the exponential bound of Hanson and Wright (1971), we obtain for some generic constant $c$:

$$S_1 \le 2\exp\Big(-c(n_r+1)\eta_n(r)\times\min\Big(\frac{1}{\max(\lambda_i)},\frac{(n_r+1)\eta_n(r)}{\sum\lambda_i^2}\Big)\Big)$$

Next, since $B^\top B$ and $BB^\top$ have the same non zero eigenvalues, we write

$$\begin{aligned}\max_{i=1,\dots,d_n}\lambda_i &\le \max_{0\le k\le n_r+1}\sum_{\ell=0}^{n_r+1}\Big|\mathbb{E}\big(\Delta_{r,k}^{(u)}X-\mathbb{E}(\Delta_{r,k}^{(u)}X)\big)\big(\Delta_{r,\ell}^{(u)}X-\mathbb{E}(\Delta_{r,\ell}^{(u)}X)\big)\Big|\\ &\le \max_{0\le k\le n_r+1}\sum_{\ell=0}^{n_r+1}|\,\mathbb{C}_r(k,\ell)|\end{aligned}$$

with $\mathbb{C}_r(k,\ell)=\mathrm{Cov}\left(\Delta_{r,k}^{(u)}X,\Delta_{r,\ell}^{(u)}X\right)$. Moreover

$$\sum_{i=1}^{d_n}\lambda_i^2=\sum_{i=1}^{d_n}\sum_{j=1}^{d_n}c_{ij}c_{ji}=\sum_{k=0}^{n_r+1}\sum_{\ell=0}^{n_r+1}(\sum_{i=1}^{d_n}b_{ki}b_{li})^2=\sum_{k=0}^{n_r+1}\sum_{\ell=0}^{n_r+1}\mathbb{C}_r^2(k,\ell).$$

Finally

(4.17)

$$S_1 \;\le\; 2\exp\Big(-c(n_r\;+\;1)\eta_n(r)\;\times\;\min\Big(\big(\max_{0\le k\le n_r+1}\sum_{\ell=0}^{n_r+1}|\,\mathbb{C}_r(k,\ell)|\big)^{-1},\frac{(n_r+1)\eta_n(r)}{\sum_{k=0}^{n_r+1}\sum_{\ell=0}^{n_r+1}\mathbb{C}_r^2(k,\ell)}\Big)\Big)$$

For the term $S_2$ in (4.16), we employ the following exponential bound with $Y\sim\mathcal{N}(0,\sigma^2)$, $\sigma>0$: for all $\varepsilon>0$,

$$\mathbb{P}\big(|Y|\ge\varepsilon\big)\le\min(1,\sqrt{\frac{2\sigma^2}{\pi\varepsilon^2}})\times\exp\big(-\frac{\varepsilon^2}{2\sigma^2}\big).$$

We set $Y=\sum_{k=0}^{n_r}(\mathbb{E}(\Delta_{r,k}^{(u)}X))\big(\Delta_{r,k}^{(u)}X-\mathbb{E}(\Delta_{r,k}^{(u)}X)\big)$ and $\varepsilon=\big(\frac{n_r+1}{4}\big)\eta_n(r)$. We obtain that $\mathrm{Var}\,(Y)\le v_n(r)$ with

$$v_n(r):=n\max_{k=0,\dots,n_r}\big(\mathbb{E}(\Delta_{r,k}^{(u)}X)\big)^2\max_{k=0,\dots,n_r}\sum_{\ell=0}^{n_r}|\mathbb{C}_r(k,\ell)| \tag{4.18}$$

so that

$$S_2\le\min\big(1,\frac{4\sqrt{2v_n(r)}}{\sqrt{\pi}(n_r+1)\eta_n(r)}\big)\exp\Big(-\frac{(n_r+1)^2\eta_n^2(r)}{32v_n(r)}\Big). \tag{4.19}$$

Finally collecting the results (4.16)-(4.19), we obtain:

(4.20)
$$\mathbb{P}\Big(\Big|\overline{(\Delta_r^{(u)}X)^2}-\mathbb{E}\Big(\overline{(\Delta_r^{(u)}X)^2}\Big)\Big|\geq\eta_n(r)\Big)=\mathcal{O}\Big(\min\big(1,\frac{\sqrt{v_n(r)}}{n\eta_n(r)}\big)\exp\big(-C(r)\frac{n^2\eta_n^2(r)}{v_n(r)}\big)\Big)$$
$$+\mathcal{O}\Big(\exp\Big(-C(r)n\eta_n(r)\times\min\Big(\big(\max_{0\le k\le n_r+1}\sum_{\ell=0}^{n_r+1}|\,\mathbb{C}_r(k,\ell)|\big)^{-1},\frac{n\eta_n(r)}{\sum\limits_{k,\ell=0}^{n_r+1}\mathbb{C}_r^2(k,\ell)}\Big)\Big)\Big)$$

for some positive constant $C(r)$, not depending on $n$ nor $\eta_n(r)$. Now from (4.20) and Lemma 4.1, one may deduce all exponential bounds claimed in (4.11)-(4.15).
(i) Case $r=r_0+1$ or $r=r_0+2$ and A2.2(1) holds. First, for $r=r_0+1$ or $r=r_0+2$, and since $\mu(.)$ is $(r_0+1)$-times continuously differentiable, a Taylor decomposition of order $r_0+1$ (similarly to the derivation of (4.3)) gives $\mathbb{E}(\Delta_{r,k}^{(u)}X)=\mathcal{O}(n^{-(r_0+1)})$. Next, from (4.18) and Lemma 4.1(i), one obtains $v_n(r)=\mathcal{O}\big(n^{-2r_0}\varphi_{1n}^{-1}(\beta_0)\big)$ with $\varphi_{1n}(\beta_0)$ given by the relation (4.9), implying in turn that $S_2=\mathcal{O}\big(\exp(-C(r)n^{2(r_0+1)}\varphi_{1n}(\beta_0))\eta_n^2(r)\big)$. Moreover, for $\psi_n(\beta_0)$ defined in (4.8) and $\psi_n(\beta_0)\eta_n(r)\to0$, we have : $S_1=\mathcal{O}\Big(\exp\big(-C(r)\varphi_{2n}(\beta_0)\eta_n^2(r)\big)\Big)$ for, again, some generic positive constant $C(r)$ and with $\varphi_{2n}(\beta_0)$ given by (4.10). As a consequence, one gets that $S_2=o(S_1)$ for all $\beta_0\in]0,\frac{3}{4}]$ while $S_2=\mathcal{O}(S_1)$ for $\beta_0\in]\frac{3}{4},1[$ (this last case is of no interest since exponentials don't converge toward 0).

On the other hand, if $\psi_n(\beta_0)\eta_n(r)\to\zeta\in]0,+\infty]$, one gets the claimed bound from (4.20) since $S_2$ is the worst of the same order as $S_1=\mathcal{O}\Big(\exp\big(-C(r)\varphi_{1n}(\beta_0)\eta_n(r)\big)\Big)$.
(ii) Case $r=r_0+2$ and Assumption A2.2(2) holds. In this case, Lemma 4.1(ii) implies that $v_n(r)=\mathcal{O}\big(n^{-(4r_0+2\beta_0+1)}\big)$ for all $\beta_0\in]0,1[$, and if $\psi_n(\beta_0)\eta_n(r)\to0$, $S_2$ is again negligible toward $S_1=\mathcal{O}\Big(\exp\big(-C(r)n^{2H+1}\eta_n^2(r)\big)\Big)$. The same conclusion holds for $\psi_n(\beta_0)\eta_n(r)\to\zeta\in]0,+\infty]$, since the bound (4.20) results in $S_1=\mathcal{O}\Big(\exp\big(-C(r)n^{H+1}\eta_n(r)\big)\Big)$.
(iii) Case $r=1,\ldots,r_0$ ($r_0\geq1$). For this case, we have $\mathbb{E}(\Delta_{r,k}^{(u)}X)=\mathcal{O}(n^{-r})$ and, Lemma 4.1(iii) implies that $\max\limits_{0\le k\le n_r+1}\sum\limits_{\ell=0}^{n_r+1}|\,\mathbb{C}_r(k,\ell)|=\mathcal{O}(n^{-2r+1})$. We get that $v_n(r)=\mathcal{O}\big(n^{-4r+2}\big)$ and $S_1=\mathcal{O}\Big(\exp\big(-C(r)n^{2r}\eta_n(r)\big)\Big)$. Next, the condition $n^{2r}\eta_n(r)\to\infty$ implies

$$\mathbb{P}\Big(\Big|\overline{(\Delta_r^{(u)}X)^2}-\mathbb{E}\Big(\overline{(\Delta_r^{(u)}X)^2}\Big)\Big|\geq\eta_n(r)\Big)$$
$$=\mathcal{O}\Big(\frac{1}{n^{2r}\eta_n(r)}\exp\big(-C(r)n^{4r}\eta_n^2(r)\big)\Big)+\mathcal{O}\Big(\exp\big(-C(r)n^{2r}\eta_n(r)\big)\Big).$$

and the first term is again negligible. □

### 4.3. **Proof of Theorem 2.1.**

*Proof.* Recall that $\widehat{r_0}$, defined by equation (2.5), is given by:

$$\widehat{r_0} = \min\Big\{r \in \{2, \ldots, m_n\} \quad : \quad B_n(r) \text{ holds}\Big\} - 2$$

where the event $B_n(r)$ is defined by $B_n(r) = \big\{n^{2r-2}\overline{\big(\Delta_r^{(1)}X\big)^2} \geq b_n\big\}$.

First remark that $m_n \to \infty$ guarantees that for $n$ large enough, $r_0 + 2 \in \{2, \ldots, m_n\}$. From the definition of $r_0$, we write

$$\mathbb{E}\big(\widehat{r_0} - r_0\big)^2 = \sum_{r=0}^{m_n-2} (r-r_0)^2 \mathbb{P}\big(\widehat{r_0} = r\big) + (l_0 - r_0)^2 \mathbb{P}\big(\widehat{r_0} = l_0\big)$$

and

$$\begin{cases} \mathbb{P}\big(\widehat{r_0} = 0\big) = \mathbb{P}\big(B_n(2)\big) \text{ if } r = 0, \\ \mathbb{P}\big(\widehat{r_0} = r\big) = \mathbb{P}\big(B_n^c(2) \cap \cdots \cap B_n^c(r+1) \cap B_n(r+2)\big) \text{ if } r = 1, \ldots, m_n - 2, \\ \mathbb{P}\big(\widehat{r_0} = l_0\big) \leq \mathbb{P}\big(B_n^c(r_0+2)\big). \end{cases}$$

Next, setting $\sum_0^{-1} \cdots = \sum_1^{-1} \cdots = \sum_1^{0} \cdots \equiv 0$, we have for all $r_0 \in \mathbb{N}_0$:

$$\begin{aligned} \mathbb{E}\big(\widehat{r_0} - r_0\big)^2 = {} & r_0^2 \mathbb{P}\big(B_n(2)\big) + \sum_{r=1}^{r_0-1} (r-r_0)^2 \mathbb{P}\big(B_n^c(2) \cap \cdots \cap B_n^c(r+1) \cap B_n(r+2)\big) \\ & + \sum_{r=r_0+1}^{m_n-2} (r-r_0)^2 \mathbb{P}\big(B_n^c(2) \cap \cdots \cap B_n^c(r+1) \cap B_n(r+2)\big) \\ & + (l_0 - r_0)^2 \mathbb{P}\big(\widehat{r_0} = l_0\big) \\ \leq {} & \sum_{r=0}^{r_0-1} (r-r_0)^2 \mathbb{P}\big(B_n(r+2)\big) + \mathbb{P}\big(B_n^c(r_0+2)\big)\Big(\sum_{r=r_0+1}^{m_n-2} (r-r_0)^2 + (l_0-r_0)^2\Big) \\ \leq {} & r_0^2 \sum_{r=0}^{r_0-1} \mathbb{P}\big(B_n(r+2)\big) + \mathcal{O}\Big(\mathbb{P}\big(B_n^c(r_0+2)\big) m_n^3\Big) \\ = {} & \mathcal{O}\big(T_{1n}(r_0)\big) + \mathcal{O}\big(m_n^3 T_{2n}(r_0)\big) \end{aligned}$$

where we have set $T_{1n}(0) = 0$ and

$$T_{1n}(r_0) = \sum_{r=2}^{r_0+1} \mathbb{P}\big(B_n(r)\big), \quad \text{for} \quad r_0 \geq 1 \tag{4.21}$$

$$T_{2n}(r_0) = \mathbb{P}\big(B_n^c(r_0+2)\big). \tag{4.22}$$

Now, the study of terms $T_{1n}$ and $T_{2n}$ may be derived from Proposition 4.1.

**Term $T_{1n}(r_0)$ (when $r_0 \geq 1$).**

If $r_1 := r_0 + 1$ and $\sum_2^1 \cdots \equiv 0$, we have

$$\begin{aligned}
T_{1n}(r_0) &= \sum_{r=2}^{r_0} \mathbb{P}\Big(\overline{(\Delta_r^{(1)}X)^2} \geq b_n n^{2-2r}\Big) + \mathbb{P}\Big(\overline{(\Delta_{r_1}^{(1)}X)^2} \geq b_n n^{-2r_0}\Big) \\
&= \sum_{r=2}^{r_0} \mathbb{P}\Big(\overline{(\Delta_r^{(1)}X)^2} - \mathbb{E}\overline{(\Delta_r^{(1)}X)^2} \geq b_n n^{2-2r} - \mathbb{E}\overline{(\Delta_r^{(1)}X)^2}\Big) \\
&\quad + \mathbb{P}\Big(\overline{(\Delta_{r_1}^{(1)}X)^2} - \mathbb{E}\overline{(\Delta_{r_1}^{(1)}X)^2} \geq b_n n^{-2r_0} - \mathbb{E}\overline{(\Delta_{r_1}^{(1)}X)^2}\Big)
\end{aligned}$$

For $r = 2, \ldots, r_0$ ($r_0 \geq 2$) and $\eta_n(r) = b_n n^{2-2r} - \mathbb{E}\overline{(\Delta_r^{(1)}X)^2}$, one obtains $n^{2r}\eta_n(r) \sim b_n n^2 \to \infty$ by relation (2.4) of Proposition 2.1(ii). We make use of the bound (4.15) of Proposition 4.1 to obtain that the first term is of order $\mathcal{O}\big(\exp(-D(r_0)\, b_n n^2)\big)$. For the second term, we set $\eta_n(r_0) = b_n n^{-2r_0} - \mathbb{E}\overline{(\Delta_{r_1}^{(1)}X)^2}$. Proposition 2.1(i) implies that $\psi_n(\beta_0)\eta_n(r_0) \geq b_n n^{2\beta_0} \to \infty$ for all $\beta_0 \in ]0,1[$, which gives, help to the exponential bound established in (4.12), a $\mathcal{O}\Big(\exp\big(-D(r_0)b_n(n^{2\beta_0+1}\mathbb{1}_{]0,\frac{1}{2}[}(\beta_0) + \frac{n^2}{\ln n}\mathbb{1}_{\{\frac{1}{2}\}}(\beta_0) + n^2\mathbb{1}_{]\frac{1}{2},1[}(\beta_0))\big)\Big)$ for this term. To conclude, we obtain that :

$$T_{1n}(r_0) = \mathcal{O}\Big(\exp\Big(-D(r_0)b_n\big(n^{2\beta_0+1}\mathbb{1}_{]0,\frac{1}{2}[}(\beta_0) + (\frac{n^2}{\ln n})\mathbb{1}_{\{\frac{1}{2}\}}(\beta_0) + n^2\mathbb{1}_{]\frac{1}{2},1[}(\beta_0)\big)\Big)\Big).$$

**Term $T_{2n}(r_0)$.** We have for $r_2 = r_0 + 2$:

$$\begin{aligned}
T_{2n}(r_0) &= \mathbb{P}\Big(\overline{\big(\Delta_{r_2}^{(1)}X\big)^2} < b_n n^{-2r_0-2}\Big) \\
&= \mathbb{P}\bigg(\mathbb{E}\Big(\overline{\big(\Delta_{r_2}^{(1)}X\big)^2}\Big) - \overline{\big(\Delta_{r_2}^{(1)}X\big)^2} > \mathbb{E}\Big(\overline{\big(\Delta_{r_2}^{(1)}X\big)^2}\Big) - b_n n^{-2r_0-2}\bigg).
\end{aligned}$$

If we set $\eta_n(r_0) = \mathbb{E}\Big(\overline{\big(\Delta_{r_2}^{(1)}X\big)^2}\Big) - b_n n^{-2r_0-2}$, the condition $b_n n^{2\beta_0-2} \to 0$ and Proposition 2.1(i) imply that $\psi_n(\beta_0)\eta_n(r_0) \geq n^H\eta_n(r_0) \to T^H\ell(r_2, r_0, \beta_0) > 0$ with $\ell(r_2, r_0, \beta_0)$ given in relation (2.3). For some positive generic constant $D(r_0)$ not depending on $n$, Proposition 4.1 with the bound (4.12) gives, if Assumption A2.2(1) holds:

$$T_{2n}(r_0) = O\bigg(\exp\Big(-D(r_0)\big(n\mathbb{1}_{]0,\frac{1}{2}[}(\beta_0) + (\frac{n}{\ln n})\mathbb{1}_{\{\frac{1}{2}\}}(\beta_0) + n^{2(1-\beta_0)}\mathbb{1}_{]\frac{1}{2},1[}(\beta_0)\big)\Big)\bigg),$$

while if Assumption A2.2(2) holds, the bound (4.14) results in:

$$T_{2n}(r_0) = O\Big(\exp(-D(r_0)n)\Big)$$

for all $\beta_0 \in ]0,1[$. For $p = 1, 2$, we get that $T_{1n}(r_0) = o\big(T_{2n}(r_0)\big)$ and the claimed result follows. $\square$

4.4. **Proof of Theorem 2.2.**

*Proof.* We start the proof with, either $p=1$ or $p=2$, and thus denote by $\widehat{r_p}$ (resp. $r_p$) the quantity $\widehat{r_0}+p$ (resp. $r_0+p$). For some $\overline{r_p}\geq r_p$, we set

$$l_n(\beta_0,r_0,p)=-\frac{1}{2}\frac{1}{(n-u\overline{r_p}+1)}\sum_{k=0}^{n-u\overline{r_p}}\sum_{i,j=0}^{\overline{r_p}}a_{i,r_p}a_{j,r_p}(ij)^{r_0} \times\iint_{[0,1]^2}\frac{(1-v)^{r_0-1}(1-w)^{r_0-1}}{(r_0-1)!^2}\left|iv-jw\right|^{2\beta_0}d_0\big((k+juw)\delta_n\big)\,\mathrm{d}v\mathrm{d}w, \tag{4.23}$$

for all $r_0\geq 1$. In the case where $r_0=0$, we set

$$l_n(\beta_0,0,p)=-\frac{1}{2}\frac{1}{(n-u\overline{r_p}+1)}\sum_{k=0}^{n-u\overline{r_p}}\sum_{i,j=0}^{\overline{r_p}}a_{i,r_p}a_{j,r_p}\left|i-j\right|^{2\beta_0}d_0(\frac{k+ju}{n}). \tag{4.24}$$

For some $\overline{\widehat{r_p}}\geq\widehat{r_p}$, we consider the following decomposition :

$$\begin{aligned}\ln(u/v)\widehat{H}_n^{(p)}&=\ln\Big(\frac{n^H}{n-u\overline{\widehat{r_p}}+1}\sum_{k=0}^{n-u\overline{\widehat{r_p}}}\big(\Delta^{(u)}_{\widehat{r_p},k}X\big)^2-(uT)^Hl_n(\beta_0,r_0,p)+(uT)^Hl_n(\beta_0,r_0,p)\Big)\\&\quad-\ln\Big(\frac{n^H}{n-v\overline{\widehat{r_p}}+1}\sum_{k=0}^{n-v\overline{\widehat{r_p}}}\big(\Delta^{(v)}_{\widehat{r_p},k}X\big)^2-(vT)^Hl_n(\beta_0,r_0,p)+(vT)^Hl_n(\beta_0,r_0,p)\Big)\end{aligned}$$

Hence

$$\ln(u/v)(\widehat{H}_n^{(p)}-H)=F_n(u)-F_n(v)+o(F_n(u)+F_n(v)) \tag{4.25}$$

with

$$F_n(u)=\frac{n^H\overline{\big(\Delta^{(u)}_{\widehat{r_p}}X\big)^2}-(uT)^Hl_n(\beta_0,r_0,p)}{(uT)^Hl_n(\beta_0,r_0,p)}=\frac{F_{1,n,p}(u)+F_{2,n,p}(u)+F_{3,n,p}(u)}{(uT)^Hl_n(\beta_0,r_0,p)}$$

where $F_{1,n,p}(u)=n^H\Big(\overline{\big(\Delta^{(u)}_{\widehat{r_p}}X\big)^2}-\overline{\big(\Delta^{(u)}_{r_p}X_k\big)^2}\Big)$, $F_{2,n,p}(u)=n^H\Bigg(\overline{\big(\Delta^{(u)}_{r_p}X\big)^2}-\mathbb{E}\Big(\overline{\big(\Delta^{(u)}_{r_p}X\big)^2}\Big)\Bigg)$ and $F_{3,n,p}(u)=n^H\mathbb{E}\Big(\overline{\big(\Delta^{(u)}_{r_p}X\big)^2}\Big)-(uT)^H\,l_n(\beta_0,r_0,p)$.

**(i)** ***Study of*** $F_{1,n,p}(u)$. We show in this part that, almost surely for $n$ large enough, $F_{1,n,p}(u)\equiv 0$ for $p=1,2$ (as a consequence of $\widehat{r_0}=r_0$, a.s. for $n$ large enough). For this, we proceed similarly as in Blanke and Vial (2011), and as previously, we set

$$B_n(r)=\big\{n^{2r-2}\overline{\big(\Delta^{(1)}_rX\big)^2}\geq b_n\big\}.$$

First, we have

$$\begin{cases}\{\widehat{r_0}=0\}=B_n(2) & \text{if } r_0=0,\\ \{\widehat{r_0}=r_0\}=B_n^c(2)\cap\cdots\cap B_n^c(r_0+1)\cap B_n(r_0+2) & \text{if } r_0\geq 1\end{cases}$$

so that

$$\begin{cases}\{\widehat{r_0}\neq r_0\}=B_n^c(2) & \text{if } r_0=0\\ \{\widehat{r_0}\neq r_0\}=B_n(2)\cup\cdots\cup B_n(r_0+1)\cup B_n^c(r_0+2) & \text{if } r_0\geq 1.\end{cases}$$

Then, for all $r_0 \geq 0$ and $n$ large enough: $\mathbb{P}(\widehat{r_0} \neq r_0) \leq T_{1n}(r_0) + T_{2n}(r_0)$ with $T_{1n}(r_0)$ and $T_{2n}(r_0)$ defined in (4.21)-(4.22). We have shown that $T_{1n} = o\big(T_{2n}(r_0)\big)$ and $T_{2n}(r_0)$ has an exponential decreasing. This implies that $\sum_n \mathbb{P}(\widehat{r_0} \neq r_0) < \infty$, so, almost surely for $n$ large enough, $\widehat{r_0} = r_0$. As a consequence:

$$F_{1,n,p}(u) \equiv 0 \tag{4.26}$$

for $p = 1$ or $p = 2$, a.s. for $n$ large enough.

**(ii)** ***Study of*** $F_{2,n,p}(u)$**.** We apply directly Proposition 4.1 with $p = 1$ or $p = 2$. For $p = 1$, we have: $\mathbb{P}\Big(\,|F_{2,n,1}(u)| \geq \varepsilon\Big) = \mathbb{P}\Big(\Big|\overline{\big(\Delta_{r_1}^{(u)} X\big)^2} - \mathbb{E}\Big(\overline{\big(\Delta_{r_1}^{(u)} X\big)^2}\Big)\Big| \geq n^{-H}\varepsilon\Big)$. Next, for $\eta_n = n^{-H}\varepsilon_{1n}(\beta_0)$ where

$$\varepsilon := \varepsilon_{1n}(\beta_0) = \varepsilon_0\Big(\frac{\ln n}{n}\Big)^{\frac{1}{2}}\mathbb{1}_{]0,\frac{3}{4}[}(\beta_0) + \varepsilon_0\Big(\frac{\ln n}{n^{1/2}}\Big)\mathbb{1}_{\{\frac{3}{4}\}}(\beta_0) + \varepsilon_0\Big(\frac{\ln n}{n^{2(1-\beta_0)}}\Big)\mathbb{1}_{]\frac{3}{4},1[}(\beta_0),$$

$\varepsilon_0 > 0$, we get that $\psi_n(\beta_0)\eta_n \to 0$ for $\beta_0 \in ]0, \frac{3}{4}[$, $\psi_n(\beta_0)\eta_n \to \zeta > 0$ if $\beta_0 = \frac{3}{4}$ while $\psi_n(\beta_0)\eta_n \to \infty$ for $\beta_0 \in ]\frac{3}{4}, 1[$. From (4.11)-(4.12), we may deduce that for all $\beta_0 \in ]0, 1[$,

$$\mathbb{P}\Big(\,|F_{2,n,1}(u)| \geq \varepsilon_{1n}(\beta_0)\Big) = \mathcal{O}\big(\exp(-D_1(r_1)\varepsilon_0^2 \ln n)\big)$$

which gives, for $\varepsilon_0$ chosen large enough, $\displaystyle\sum_n \mathbb{P}\big(\,|F_{2,n,1}(u)| \geq \varepsilon_{1n}(\beta_0)\big) < +\infty$ so that, almost surely,

$$\limsup_{n\to\infty} \varepsilon_{1n}^{-1}(\beta_0)\,|F_{2,n,1}(u)| < +\infty. \tag{4.27}$$

For $p = 2$, one may apply the bound (4.13) with $\varepsilon_{2n} = \varepsilon_0\Big(\dfrac{\ln n}{n}\Big)^{\frac{1}{2}}$ to get the same result for all $\beta_0 \in ]0, 1[$ and $\varepsilon_0$ large enough: almost surely,

$$\limsup_{n\to\infty} \varepsilon_{2n}^{-1}\,|F_{2,n,2}(u)| < +\infty. \tag{4.28}$$

**(iii) *Study of* $F_{3,n,p}(u)$.** From (4.3) and proceeding similarly as in (4.4) in the case $r_0 \geq 1$, we get

$$
\begin{aligned}
& n^H \mathbb{E}\Big( \overline{\big(\Delta_{r_p}^{(u)} X\big)^2} \Big) \sim T^H \delta_n^{-H} \mathbb{E}\Big( \overline{\big(\Delta_{r_p}^{(u)} X\big)^2} \Big) \\
& = T^H \sum_{k=0}^{n-u\overline{r_p}} \sum_{i,j=0}^{\overline{r_p}} \delta_n^{-2\beta_0} \frac{a_{i,r_p} a_{j,r_p} (u^2 ij)^{r_0}}{n-u\overline{r_p}+1} \iint_{[0,1]^2} \frac{\big((1-v)(1-w)\big)^{r_0-1}}{((r_0-1)!)^2} \mathbb{L}^{(r_0,r_0)}\big(\dot{k}_{iuv}, \dot{k}_{juw}\big) \mathrm{d}v\mathrm{d}w \\
& = (uT)^H l_n(\beta_0, r_0, p) - (uT)^H \sum_{k=0}^{n-u\overline{r_p}} \sum_{i,j=0}^{\overline{r_p}} \frac{a_{i,r_p} a_{j,r_p} (ij)^{r_0}}{2(n-u\overline{r_p}+1)} \iint_{[0,1]^2} \frac{\big((1-v)(1-w)\big)^{r_0-1}}{((r_0-1)!)^2} \\
& \qquad \times |iv - jw|^{2\beta_0} \Bigg\{ \frac{\mathbb{L}^{(r_0,r_0)}(\dot{k}_{iuv}, \dot{k}_{iuv}) + \mathbb{L}^{(r_0,r_0)}\big(\dot{k}_{juw}, \dot{k}_{juw}\big) - 2\mathbb{L}^{(r_0,r_0)}\big(\dot{k}_{iuv}, \dot{k}_{juw}\big)}{\big|\dot{k}_{iuv} - \dot{k}_{juw}\big|^{2\beta_0}} \\
& \qquad\qquad - d_0(\dot{k}_{juw}) \Bigg\} \mathrm{d}v\mathrm{d}w
\end{aligned}
$$

with $l_n(\beta_0, r_0, p)$ given by (4.23). Next, we introduce condition (2.6) by adding and subtracting the function $d_1$:

$$
\begin{aligned}
& n^{\beta_1} \bigg( n^H \mathbb{E}\Big( \overline{\big(\Delta_{r_p} X^{(u)}\big)^2} \Big) - (uT)^H l_n(\beta_0, r_0, p) \bigg) = \\
& - \frac{(uT)^{H+\beta_1}}{2} \sum_{k=0}^{n-u\overline{r_p}} \sum_{i,j=0}^{\overline{r_p}} \frac{a_{i,r_p} a_{j,r_p} (ij)^{r_0}}{(n-u\overline{r_p}+1)} \iint_{[0,1]^2} \frac{\big((1-v)(1-w)\big)^{r_0-1}}{((r_0-1)!)^2} |iv - jw|^{2\beta_0+\beta_1} \\
& \times \left\{ \frac{\frac{\mathbb{L}^{(r_0,r_0)}(\dot{k}_{iuv}, \dot{k}_{iuv}) + \mathbb{L}^{(r_0,r_0)}\big(\dot{k}_{juw}, \dot{k}_{juw}\big) - 2\mathbb{L}^{(r_0,r_0)}\big(\dot{k}_{iuv}, \dot{k}_{juw}\big)}{\big|\dot{k}_{iuv} - \dot{k}_{juw}\big|^{2\beta_0}} - d_0(\dot{k}_{juw})}{\big|\dot{k}_{iuv} - \dot{k}_{juw}\big|^{\beta_1}} - d_1(\dot{k}_{juw}) \right\} \mathrm{d}v\mathrm{d}w \\
& - \frac{(uT)^{H+\beta_1}}{2} \sum_{k=0}^{n-u\overline{r_p}} \sum_{i,j=0}^{\overline{r_p}} \frac{a_{i,r_p} a_{j,r_p} (ij)^{r_0}}{(n-u\overline{r_p}+1)} \iint_{[0,1]^2} \frac{\big((1-v)(1-w)\big)^{r_0-1}}{((r_0-1)!)^2} |iv - jw|^{2\beta_0+\beta_1} \\
& \qquad\qquad d_1(\dot{k}_{juw}) \, \mathrm{d}v\mathrm{d}w.
\end{aligned}
$$

The condition (2.6) implies the convergence of the inner braces to 0 (uniformly in $v$, $w$, $k$), so for $p = 1$ or $p = 2$, the uniform continuity of function $d_1$ gives:

$$
\begin{aligned}
n^{\beta_1} F_{3,n,p}(u) \xrightarrow[n\to\infty]{} & -\frac{(uT)^{H+\beta_1}}{2} \sum_{i,j=0}^{\overline{r_p}} a_{i,r_p} a_{j,r_p} (ij)^{r_0} \\
& \times \iint_{[0,1]^2} \frac{\big((1-v)(1-w)\big)^{r_0-1}}{((r_0-1)!)^2} |iv - jw|^{2\beta_0+\beta_1} \, \mathrm{d}v\mathrm{d}w \times \Big(\frac{1}{T} \int_0^T d_1(t) \, \mathrm{d}t\Big).
\end{aligned}
\tag{4.29}
$$

Finally, the case $r_0 = 0$ is treated similarly, starting from (4.2) instead of (4.3).

**Conclusion.** To conclude, one may note that the denominator term, $l_n(\beta_0, r_0, p)$, defined in (4.23)-(4.24), converges to the nonzero term:

$$\begin{aligned} l_n(\beta_0, r_0, p) \xrightarrow[n\to\infty]{} -\frac{1}{2}\sum_{i,j=0}^{\overline{r_p}} a_{i,r_p}a_{j,r_p}(ij)^{r_0} \\ \times \iint_{[0,1]^2} \frac{(1-v)^{r_0-1}(1-w)^{r_0-1}}{(r_0-1)!^2}\left|iv-jw\right|^{2\beta_0}\,\mathrm{d}v\mathrm{d}w\frac{1}{T}\int_0^T d_0(t)\,\mathrm{d}t \end{aligned}$$

for $r_0 \geq 1$ while if $r_0 = 0$, $l_n(\beta_0, r_0, p) \xrightarrow[n\to\infty]{} -\frac{1}{2}\sum_{i,j=0}^{\overline{r_p}} a_{i,r_p}a_{j,r_p}\left|i-j\right|^{2\beta_0}\frac{1}{T}\int_0^T d_0(t)\,\mathrm{d}t$.

Relation (4.25) and expressions established in (4.26)-(4.29) lead to the final result. □

### 4.5. **Proof of Theorem 3.1.**

*Proof.* First, we set $\widetilde{r_0} = \max(\widehat{r_0}, 1)$ and, for $\widehat{r_0}$ and $\widetilde{X}_r(\cdot)$ respectively defined in (2.5) and (3.1), we adopt the simpler convention that $\widehat{r_0} = l_0 \Rightarrow \widetilde{X}_{\widetilde{r_0}}(\cdot) = \widetilde{X}_{m_n-1}(\cdot)$ and $\widetilde{X}_{\widehat{r_0}+1}(\cdot) = \widetilde{X}_{m_n}(\cdot)$. Also, we will make use of the following lemma:

**Lemma 4.2.** *For any Gaussian variable $Y$, $\mathbb{E}(Y^4) \leq 3\big(\mathbb{E}(Y^2)\big)^2$.*

4.5.1. *Study of $e_\rho^2(\mathrm{app}(\widehat{\mathrm{r}_0}))$.* We may write

$$\begin{aligned} \big(X(t) - \widetilde{X}_{\widetilde{r_0}}(t)\big)^2 &= \sum_{r=0}^{m_n-2}\big(X(t) - \widetilde{X}_{\widetilde{r_0}}(t)\big)^2\mathbb{1}_{\{\widehat{r_0}=r\}} + \big(X(t) - \widetilde{X}_{\widetilde{r_0}}(t)\big)^2\mathbb{1}_{\{\widehat{r_0}=l_0\}} \\ &= \sum_{r=0}^{m_n-2}\big(X(t) - \widetilde{X}_{\overline{r}}(t)\big)^2\mathbb{1}_{\{\widehat{r_0}=r\}} + \big(X(t) - \widetilde{X}_{m_n-1}(t)\big)^2\mathbb{1}_{\{\widehat{r_0}=l_0\}} \end{aligned}$$

where $\overline{r} = \max(r, 1)$ and for $\overline{r_0} = \max(r_0, 1)$,

$$\big(X(t) - \widetilde{X}_{\widetilde{r_0}}(t)\big)^2 \leq \big(X(t) - \widetilde{X}_{\overline{r_0}}(t)\big)^2 + \mathbb{1}_{\{\widehat{r_0}\neq r_0\}}\sum_{r=0,r\neq r_0}^{m_n-1}\big(X(t) - \widetilde{X}_{\overline{r}}(t)\big)^2$$

Using Cauchy-Schwarz inequality:

$$\begin{aligned} e_\rho^2(\mathrm{app}(\widehat{r_0})) \leq \int_0^{\mathrm{T}} \mathbb{E}\big(\mathrm{X}(\mathrm{t}) - \widetilde{\mathrm{X}}_{\overline{\mathrm{r_0}}}(\mathrm{t})\big)^2\rho(\mathrm{t})\,\mathrm{dt} \\ + \big(\mathbb{P}(\widehat{r_0}\neq r_0)\big)^{\frac{1}{2}}\sum_{r=0,r\neq r_0}^{m_n-1}\int_0^T\Big(\mathbb{E}\big(X(t) - \widetilde{X}_{\overline{r}}(t)\big)^4\Big)^{\frac{1}{2}}\rho(t)\,\mathrm{d}t \end{aligned}$$

turning to, by lemma 4.2,

$$e_\rho^2\big(\mathrm{app}(\widehat{\mathrm{r}_0})\big) \le \sup_{\mathrm{t}\in[0,\mathrm{T}]}\Big(\mathbb{E}\big(\mathrm{X}(\mathrm{t})-\widetilde{\mathrm{X}}_{\overline{\mathrm{r}_0}}(\mathrm{t})\big)^2\Big)\int_0^{\mathrm{T}}\rho(\mathrm{t})\,\mathrm{dt}$$
$$+\sqrt{3}\big(\mathbb{P}(\widehat{r_0}\neq r_0)\big)^{\frac12}\sum_{r=0,r\neq r_0}^{m_n-1}\sup_{t\in[0,T]}\Big(\mathbb{E}\big(X(t)-\widetilde{X}_{\overline{r}}(t)\big)^2\Big)\int_0^T\rho(t)\,\mathrm{d}t.$$

Study of the term $\sup_{t\in[0,T]}\Big(\mathbb{E}\big(X(t)-\widetilde{X}_r(t)\big)^2\Big)$, $r=1,\dots,m_n$. We have

$$\sup_{t\in[0,T]}\Big(\mathbb{E}\big(X(t)-\widetilde{X}_r(t)\big)^2\Big)=\max_{k=0,\dots,\lfloor\frac{n}{r}\rfloor-1}\sup_{t\in\mathcal{I}_k}\Big(\mathbb{E}\big(X(t)-\widetilde{X}_r(t)\big)^2\Big).$$

First, suppose that $r_0\ge 1$. We use the decomposition established in Blanke and Vial (2008, lemma 4.1) to obtain, for $t\in\mathcal{I}_k$ and $r^*=\min(r,r_0)$:

$$\mathbb{E}\big(X(t)-\widetilde{X}_r(t)\big)^2=\sum_{i,j=0}^{r}L_{i,k,r}(t)L_{j,k,r}(t)\frac{(ij\delta_n^2)^{r^*}}{((r^*-1)!)^2}\iint_{[0,1]^2}\big((1-v)(1-w)\big)^{r^*-1}$$
$$\times\Big\{\mathbb{L}^{(r^*,r^*)}(kr\delta_n+(t-kr\delta_n)v,kr\delta_n+(t-kr\delta_n)w)-\mathbb{L}^{(r^*,r^*)}(kr\delta_n+(t-kr\delta_n)v,kr\delta_n+j\delta_nw)$$
$$-\mathbb{L}^{(r^*,r^*)}(kr\delta_n+i\delta_nv,kr\delta_n+(t-kr\delta_n)w)+\mathbb{L}^{(r^*,r^*)}(kr\delta_n+i\delta_nv,kr\delta_n+j\delta_nw)\Big\}\,\mathrm{d}v\mathrm{d}w.$$

For $r=1,\dots,r_0-1$, ($r_0\ge 2$), we get that

$$\mathbb{E}\big(X(t)-\widetilde{X}_r(t)\big)^2=\sum_{i,j=0}^{r}L_{i,k,r}(t)L_{j,k,r}(t)\frac{(ij\delta_n^2)^{r}}{((r-1)!)^2}\iint_{[0,1]^2}\big((1-v)(1-w)\big)^{r-1}$$
$$\int_{kr\delta_n+i\delta_nv}^{kr\delta_n+(t-kr\delta_n)v}\int_{kr\delta_n+j\delta_nw}^{kr\delta_n+(t-kr\delta_n)w}\mathbb{L}^{(r+1,r+1)}(s,t)\,\mathrm{d}s\mathrm{d}t.$$

As $\mathbb{L}^{(r+1,r+1)}(\cdot,\cdot)$ is continuous, we obtain the bound: $\mathbb{E}\big(X(t)-\widetilde{X}_r(t)\big)^2=\mathcal{O}\big(\delta_n^{2r+2}\big)$. For $r=r_0,\dots,m_n$, we arrive at

$$\mathbb{E}\big(X(t)-\widetilde{X}_r(t)\big)^2=\sum_{i,j=0}^{r}L_{i,k,r}(t)L_{j,k,r}(t)\frac{(ij\delta_n^2)^{r_0}}{((r_0-1)!)^2}\int_{[0,1]^2}\big((1-v)(1-w)\big)^{r_0-1}$$
$$\times\Big\{\mathbb{L}^{(r_0,r_0)}(kr\delta_n+(t-kr\delta_n)v,kr\delta_n+(t-kr\delta_n)w)-\mathbb{L}^{(r_0,r_0)}(kr\delta_n+(t-kr\delta_n)v,kr\delta_n+j\delta_nw)$$
$$-\mathbb{L}^{(r_0,r_0)}(kr\delta_n+i\delta_nv,kr\delta_n+(t-kr\delta_n)w)+\mathbb{L}^{(r_0,r_0)}(kr\delta_n+i\delta_nv,kr\delta_n+j\delta_nw)\Big\}\,\mathrm{d}v\mathrm{d}w.$$

Adding and subtracting terms like $\frac12\mathbb{L}^{(r_0,r_0)}(kr\delta_n+(t-kr\delta_n)v,kr\delta_n+(t-kr\delta_n)v)$ and $\frac12\mathbb{L}^{(r_0,r_0)}(kr\delta_n+(t-kr\delta_n)w,kr\delta_n+(t-kr\delta_n)w)$, or $\frac12\mathbb{L}^{(r_0,r_0)}(kr\delta_n+(t-kr\delta_n)v,kr\delta_n+(t-kr\delta_n)v)$ and $\frac12\mathbb{L}^{(r_0,r_0)}(kr\delta_n+j\delta_nv,kr\delta_n+j\delta_nv)$, ..., we make use four times of the Hölderian regularity condition (4.1) to obtain, using the bound $L_{i,k,r}(t)\le r^r$, that

- for $r=r_0$, $\sup_{t\in[0,T]}\mathbb{E}\big(X(t)-\widetilde{X}_{\overline{r_0}}(t)\big)^2=\mathcal{O}\big(\delta_n^{2(r_0+\beta_0)}\big)$,

- for $r = r_0 + 1, \dots, m_n$, $\sup_{t\in[0,T]} \mathbb{E}\big(X(t) - \widetilde{X}_{\overline{r}}(t)\big)^2 = \mathcal{O}\big(m_n^{2(m_n+r_0+\beta_0)}\delta_n^{2(r_0+\beta_0)}\big)$.

Next, for $r_0 = 0$, we observe that above results hold true starting from

$$\mathbb{E}\big(X(t) - \widetilde{X}_{\overline{r}}(t)\big)^2 = \sum_{i,j=0}^{\overline{r}} L_{i,k,\overline{r}}(t)L_{j,k,\overline{r}}(t)\Big\{\mathbb{L}(t,t) - \mathbb{L}(t, k\overline{r}\delta_n + j\delta_n) - \mathbb{L}(k\overline{r}\delta_n + i\delta_n, t) + \mathbb{L}(k\overline{r}\delta_n + i\delta_n, k\overline{r}\delta_n + j\delta_n)\Big\}.$$

Collecting these result, and using the exponential upper bound established in Blanke and Vial (2011) for $\mathbb{P}(\widehat{r_0} \neq r_0)$, we arrive at the final result, help to the logarithmic order of $m_n$.

4.5.2. *Study of* $e_\rho^2(\mathrm{int}(\widehat{\mathrm{r}_0}))$. Since $\int_0^T \big(X(t) - \widetilde{X}_{r+1}\big)\rho(t)\,\mathrm{d}t$ is again a Gaussian variable, in a similar way as for approximation, we get the bound:

$$e_\rho^2(\mathrm{int}(\widehat{\mathrm{r}_0})) \leq \sum_{\mathrm{k}=0}^{\lfloor\frac{\mathrm{n}}{\mathrm{r}_0+1}\rfloor-1}\ \sum_{\ell=0}^{\lfloor\frac{\mathrm{n}}{\mathrm{r}_0+1}\rfloor-1} \int_{\mathcal{I}_\mathrm{k}}\int_{\mathcal{I}_\ell} \mathbb{E}\big(\mathrm{X(t)} - \widetilde{\mathrm{X}}_{\mathrm{r}_0+1}(\mathrm{t})\big)\big(\mathrm{X(s)} - \widetilde{\mathrm{X}}_{\mathrm{r}_0+1}(\mathrm{s})\big)\rho(\mathrm{t})\rho(\mathrm{s})\,\mathrm{dsdt}$$
$$+ \sqrt{3}\big(\mathbb{P}(\widehat{r_0} \neq r_0)\big)^{\frac{1}{2}} \sum_{r=0, r\neq r_0}^{m_n} \Big(\sup_{t\in[0,T]} \Big(\mathbb{E}\big(X(t) - \widetilde{X}_{r+1}(t)\big)^2\Big)^{\frac{1}{2}}\Big)^2 \Big(\int_0^T \rho(t)\,\mathrm{d}t\Big)^2.$$

Study of the term $\mathbb{E}\big(X(t) - \widetilde{X}_{r_0+1}(t)\big)\big(X(s) - \widetilde{X}_{r_0+1}(s)\big)$, $(s,t) \in \mathcal{I}_\ell \times \mathcal{I}_k$. Again from lemma 4.1 of Blanke and Vial (2008), we get

$$\begin{aligned}
&\mathbb{E}\big(X(t) - \widetilde{X}_{r_0+1}(t)\big)\big(X(s) - \widetilde{X}_{r_0+1}(s)\big) \\
&\quad= \sum_{i,j=0}^{r_0+1} L_{i,k,r_0+1}(t)L_{j,k,r_0+1}(s)\frac{(ij\delta_n^2)^{r_0}}{((r_0-1)!)^2}\iint_{[0,1]^2}((1-v)(1-w))^{r_0-1} \\
&\quad\times \Big\{\mathbb{L}^{(r_0,r_0)}(k(r_0+1)\delta_n + (t - k(r_0+1)\delta_n)v, \ell(r_0+1)\delta_n + (t - \ell(r_0+1)\delta_n)w) \\
&\qquad - \mathbb{L}^{(r_0,r_0)}(k(r_0+1)\delta_n + (t - k(r_0+1)\delta_n)v, \ell(r_0+1)\delta_n + j\delta_n w) \\
&\qquad - \mathbb{L}^{(r_0,r_0)}(k(r_0+1)\delta_n + i\delta_n v, \ell(r_0+1)\delta_n + (t - \ell(r_0+1)\delta_n)w) \\
&\qquad + \mathbb{L}^{(r_0,r_0)}(k(r_0+1)\delta_n + i\delta_n v, \ell(r_0+1)\delta_n + j\delta_n v)\Big\}
\end{aligned}$$

For non-overlapping intervals $\mathcal{I}_k$ and $\mathcal{I}_\ell$, that is $|k-l| \geq 2$, we make use of Condition A2.2(2) four times, by adding and subtracting the necessary terms, noting that

$$\sum_{i,j=0}^{r_0+1} L_{i,k,r_0+1}(t)L_{j,\ell,r_0+1}(s)(i\delta_n)^{r_1}(j\delta_n)^{r_2} = (t - k(r_0+1)\delta_n)^{r_1}(s - \ell(r_0+1)\delta_n)^{r_2}$$

with either $r_i = r_0$ or $r_i = r_0 + 1$ for $i = 1, 2$. By this way, we get

$$\sum_{\substack{k,\ell=0\\|k-\ell|\geq 2}}^{\lfloor \frac{n}{r_0+1}\rfloor -1} \int_{\mathcal{I}_k}\int_{\mathcal{I}_\ell} \mathbb{E}\big(X(t)-\widetilde{X}_{r_0+1}(t)\big)\big(X(s)-\widetilde{X}_{r_0+1}(s)\big)\rho(t)\rho(s)\,\mathrm{d}s\mathrm{d}t$$

$$= \mathcal{O}\Big(\delta_n^{2(r_0+\beta_0+1)} \sum_{\substack{k,\ell=0\\|k-\ell|\geq 2}}^{\lfloor \frac{n}{r_0+1}\rfloor -1} \big|\,|k-\ell|-1\big|^{-2(2-\beta_0)}\Big) = \mathcal{O}\Big(n\delta_n^{2(r_0+\beta_0+1)}\Big) = \mathcal{O}\Big(\delta_n^{2(r_0+\beta_0)+1}\Big).$$

For overlapping intervals $\mathcal{I}_k$ and $\mathcal{I}_\ell$, that is $|k-l| \leq 1$, we make use of Cauchy-Schwarz inequality to obtain the same bound as above. Since the second part of $e_\rho^2(\mathrm{int}(\widehat{\mathrm{r}_0}))$ is negligible, we obtain the claimed result. □

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